Attempt all questions and show all your work. Due October 9, 2015.

1. Simplify $\frac{169}{5+12i} + \left(\overline{(1-2i)^3+4}\right)^2$ and express in Cartesian form.

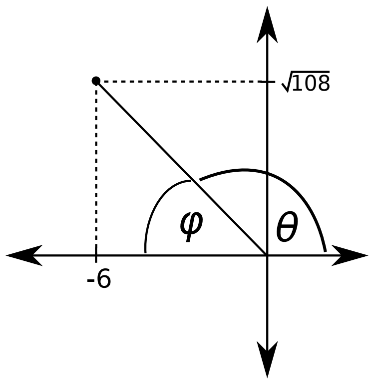
Solution:

$$\begin{aligned}
\frac{169}{5+12i} + \left(\overline{(1-2i)^3+4}\right)^2 &= \frac{169}{5+12i} + \left(\overline{(1-4i+4i^2)(1-2i)+4}\right)^2 \\
&= \frac{169}{5+12i} + \left(\overline{(-3-4i)(1-2i)+4}\right)^2 \\
&= \frac{169}{5+12i} + \left(\overline{-3+6i-4i+8i^2+4}\right)^2 \\
&= \frac{169}{5+12i} + \left(\overline{-11+2i+4}\right)^2 \\
&= \frac{169}{5+12i} + \left(\overline{-7+2i}\right)^2 \\
&= \frac{169}{5+12i}\left(\frac{5-12i}{5-12i}\right) + (-7-2i)^2 \\
&= \frac{169(5-12i)}{25-144i^2} + (49+28i+4i^2) \\
&= \frac{169(5-12i)}{169} + 45 + 28i \\
&= 5 - 12i + 45 + 28i \\
&= 50 + 16i.
\end{aligned}$$

2. Express in the forms required, with all arguments in your answers reduced to numbers in the interval $(-\pi, \pi]$.

   (a) $-6 + i\sqrt{108}$ in polar and exponential forms

   Solution:



Let $z = -6 + i\sqrt{108}$.

$r^2 = (-6)^2 + (\sqrt{108})^2 = 36 + 108 = 144$ and so $r = 12$.

$\sin\phi = \frac{\sqrt{108}}{12} = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2}$. Therefore $\phi = \frac{\pi}{3}$, and thus $\theta = \pi - \phi = \frac{2\pi}{3}$.

Polar form: $z = 12\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

Exponential form: $z = 12e^{\frac{2\pi}{3}i}$.

(b) $\sqrt{18}\left(\cos\frac{19\pi}{4} + i\sin\frac{19\pi}{4}\right)$ in Cartesian and exponential forms

Solution: Let $z = \sqrt{18}\left(\cos\frac{19\pi}{4} + i\sin\frac{19\pi}{4}\right)$.

Exponential form: $z = \sqrt{18}e^{\frac{19\pi}{4}i}$.

Cartesian form:

$$\begin{aligned} z &= \sqrt{18}\left(\cos\frac{19\pi}{4} + i\sin\frac{19\pi}{4}\right) \\ &= \sqrt{18}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \\ &= \sqrt{18}\left(\frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) \\ &= \frac{-\sqrt{36}}{2} + i\frac{\sqrt{36}}{2} \\ &= \frac{-6}{2} + i\frac{6}{2} \\ &= -3 + 3i. \end{aligned}$$

(c) $10e^{\frac{-5\pi}{6}i}$ in Cartesian and polar forms.

Solution:

$$\begin{aligned} 10e^{\frac{-5\pi}{6}i} &= 10(\cos\frac{-5\pi}{6} + i\sin\frac{-5\pi}{6}) && \text{(Polar form)} \\ &= 10(\frac{-\sqrt{3}}{2} + \frac{-1}{2}i) \\ &= -5\sqrt{3} - 5i. && \text{(Cartesian form)} \end{aligned}$$

3. $\cos n\theta$, $n \in \mathbb{Z}$, can always be expressed in terms of $\sin\theta$ and $\cos\theta$. For example, $\cos 3\theta =$

$\cos^3\theta - 3\cos\theta\sin^2\theta$. Use De Moivre's Theorem to obtain an expression of this type for $\cos 7\theta$.

Solution: Let $z = (\cos\theta + i\sin\theta)$. Then

$$\begin{aligned}
z^7 &= \cos 7\theta + i\sin 7\theta \qquad \text{by De Moivre's Theorem, and}\\
z^7 &= (\cos\theta + i\sin\theta)^7\\
&= \sum_{k=0}^{7}\binom{7}{k}\cos^{7-k}\theta\,(i\sin\theta)^k\\
&= \binom{7}{0}\cos^7\theta + \binom{7}{1}\cos^6\theta(i\sin\theta)^1 + \binom{7}{2}\cos^5\theta(i\sin\theta)^2 + \binom{7}{3}\cos^4\theta(i\sin\theta)^3\\
&\quad + \binom{7}{4}\cos^3\theta(i\sin\theta)^4 + \binom{7}{5}\cos^2\theta(i\sin\theta)^5 + \binom{7}{6}\cos\theta(i\sin\theta)^6 + \binom{7}{7}(i\sin\theta)^7\\
&= \cos^7\theta + 7i\cos^6\theta\sin\theta + 21i^2\cos^5\theta\sin^2\theta + 35i^3\cos^4\theta\sin^3\theta + 35i^4\cos^3\theta\sin^4\theta\\
&\quad + 21i^5\cos^2\theta\sin^5\theta + 7i^6\cos\theta\sin^6\theta + i^7\sin^7\theta\\
&= \cos^7\theta + 7i\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - 35i\cos^4\theta\sin^3\theta + 35\cos^3\theta\sin^4\theta\\
&\quad + 21i\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - i\sin^7\theta\\
&= \left(\cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta\right)\\
&\quad + i\left(7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta\right).
\end{aligned}$$

Therefore, equating the real and complex parts of z^7 calculated in each of the two ways, we have:

$$\begin{aligned}
\cos 7\theta &= \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta\\
\sin 7\theta &= 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta.
\end{aligned}$$

4. Find all of the complex 6th roots of -64. Express your answers in Cartesian form.

Solution: Let z be such that $z^6 = -64 = 64e^{\pi i} = 64e^{(2k+1)\pi i}$. Then

$$z = \left(64e^{(2k+1)\pi i}\right)^{1/6} = 2e^{\frac{(2k+1)\pi}{6}i}.$$

$$\begin{aligned}
k = 0 : z &= 2e^{\frac{\pi}{6}i} = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i.\\
k = 1 : z &= 2e^{\frac{\pi}{2}i} = 2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 2(0 + i) = 2i.\\
k = 2 : z &= 2e^{\frac{5\pi}{6}i} = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}) = 2(\frac{-\sqrt{3}}{2} + i\frac{1}{2}) = -\sqrt{3} + i.\\
k = 3 : z &= 2e^{\frac{7\pi}{6}i} = 2(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}) = 2(\frac{-\sqrt{3}}{2} + i\frac{-1}{2}) = -\sqrt{3} - i.
\end{aligned}$$

$$k=4: z=2e^{\frac{3\pi}{2}i}=2(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2})=2(0+i(-1))=-2i.$$
$$k=5: z=2e^{\frac{11\pi}{6}i}=2(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6})=2(\frac{\sqrt{3}}{2}+i\frac{-1}{2})=\sqrt{3}-i.$$

5. Solve the equation $x^4-8x^2+36=0$ over the complex numbers.

Solution: We will need two trig identity throughout:

$$\cos\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}} \qquad \sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}$$

Let $z=x^2$. Then we have that $z^2-8z+36=0$, which we can solve using the quadratic formula:

$$z=\frac{8\pm\sqrt{64-4(1)(36)}}{2(1)}=\frac{8\pm\sqrt{-80}}{2}=4\pm i\sqrt{20}.$$
$$|z|=\sqrt{4^2+\sqrt{20}^2}=\sqrt{16+20}=\sqrt{36}=6.$$
$$\text{Thus } z=4\pm i\sqrt{20}=6\left(\frac{4}{6}\pm i\frac{\sqrt{20}}{6}]\right)=6\left(\frac{2}{3}\pm i\frac{\sqrt{5}}{3}\right).$$

Let $z_1=4+i\sqrt{20}$, $z_2=4-i\sqrt{20}$.

- Let $\theta_1=\arg(z_1)$. First we will find x such that

$$x^2=z_1=4+i=6\left(\frac{2}{3}+i\frac{\sqrt{5}}{3}\right)=6(\cos\theta_1+i\sin\theta_1)=6e^{(\theta_1+2k\pi)i}.$$

Then:

$$x=(6e^{(\theta_1+2k\pi)i})^{1/2}=\sqrt{6}e^{(\frac{\theta_1}{2}+k\pi)i}.$$

$$\begin{aligned}
k=0: \quad x_1 &= \sqrt{6}e^{(\frac{\theta_1}{2})i}\\
&= \sqrt{6}\left(\cos\frac{\theta_1}{2}+i\sin\frac{\theta_1}{2}\right)\\
&= \sqrt{6}\left(\sqrt{\frac{1+\cos\theta_1}{2}}+i\sqrt{\frac{1-\cos\theta_1}{2}}\right) \qquad \text{(trig identities)}\\
&= \sqrt{6}\left(\sqrt{\frac{1+(\frac{2}{3})}{2}}+i\sqrt{\frac{1-(\frac{2}{3})}{2}}\right)\\
&= \sqrt{6}\left(\sqrt{\frac{5}{6}}+i\sqrt{\frac{1}{6}}\right)=\sqrt{5}+i\\
k=1: \quad x_2 &= \sqrt{6}e^{(\frac{\theta_1}{2}+\pi)i}=-\sqrt{6}e^{\frac{\theta_1}{2}i}=-\sqrt{5}-i.
\end{aligned}$$

- Let $\theta_2 = \arg(z_2)$. Now we will find x such that

$$x^2 = z_2 = 4 - i = 6\left(\frac{2}{3} - i\frac{\sqrt{5}}{3}\right) = 6(\cos\theta_2 + i\sin\theta_2) = 6e^{(\theta_2+2k\pi)i}.$$

$$x = (6e^{(\theta_2+2k\pi)i})^{1/2} = \sqrt{6}e^{(\frac{\theta_2}{2}+k\pi)i}.$$

$$\begin{aligned}
k=0: \quad x_3 &= \sqrt{6}e^{(\frac{\theta_2}{2})i} \\
&= \sqrt{6}\left(\cos\frac{\theta_2}{2} + i\sin\frac{\theta_2}{2}\right) \\
&= \sqrt{6}\left(\sqrt{\frac{1+\cos\theta_2}{2}} - i\sqrt{\frac{1-\cos\theta_2}{2}}\right) \quad \text{(trig identities and since } \sin\theta_2 < 0\text{)} \\
&= \sqrt{6}\left(\sqrt{\frac{1+(\frac{2}{3})}{2}} - i\sqrt{\frac{1-(\frac{2}{3})}{2}}\right) \\
&= \sqrt{6}\left(\sqrt{\frac{5}{6}} - i\sqrt{\frac{1}{6}}\right) = \sqrt{5} - i \\
k=1: \quad x_4 &= \sqrt{6}e^{(\frac{\theta_2}{2}+\pi)i} = -\sqrt{6}e^{\frac{\theta_2}{2}i} = -\sqrt{5} + i.
\end{aligned}$$

Therefore the four different roots of this polynomial equation are

$$\sqrt{5} + i, \quad -\sqrt{5} - i, \quad \sqrt{5} - i, \quad -\sqrt{5} + i.$$

6. (a) Use long division to find the quotient and remainder when $x^5 - 3x^4 + 2x^2 - x + 7$ is divided by $x - 3$. Express the result as an equation of the form

$$\text{(polynomial)} = \text{(polynomial)} \cdot \text{(quotient)} + \text{(remainder)}.$$

Solution:

$$\begin{array}{r|l}
 & \quad x^4 + 2x + 5 \\
\hline
x - 3 & x^5 - 3x^4 + 2x^2 - x + 7 \\
 & \underline{x^5 - 3x^4} \\
 & \qquad 0 + 2x^2 - x \\
 & \qquad\quad \underline{2x^2 - 6x} \\
 & \qquad\qquad\quad 5x + 7 \\
 & \qquad\qquad\quad \underline{5x - 15} \\
 & \qquad\qquad\qquad 22
\end{array}$$

Therefore,

$$x^5 - 3x^4 + 2x^2 - x + 7 = (x-3)(x^4 + 2x + 5) + 22$$

(b) Use the Remainder Theorem to find the remainder when

$$f(x) = (1+i)x^4 + 3ix^3 + (1-i)x + 2$$

is divided by $ix - 3$ (Do not perform long division!)

Solution: First note that

$$\frac{3}{i} = \frac{3}{i}\left(\frac{-i}{-i}\right) = \frac{-3i}{-i^2} = \frac{-3i}{1} = -3i.$$

$$\begin{aligned}
f(3/i) &= f(-3i) \\
&= (1+i)(-3i)^4 + 3i(-3i)^3 + (1-i)(-3i) + 2 \\
&= (1+i)(81) + 3i(-27i^3) - 3i + 3i^2 + 2 \\
&= 81 + 81i - 81i^4 - 3i - 3 + 2 \\
&= 81 - 81 - 3 + 2 + i(81 - 3) \\
&= -1 + 78i.
\end{aligned}$$

(c) For which value of d is the polynomial $2x - 3$ a factor of the polynomial $g(x) = x^3 - 5x^2 + 2x - d$?

Solution: We need $g(3/2) = 0$. So, we have

$$\begin{aligned}
g(3/2) = \left(\frac{3}{2}\right)^3 - 5\left(\frac{3}{2}\right)^2 + 2\left(\frac{3}{2}\right) - d &= 0 \\
\frac{27}{8} - \frac{45}{4}\left(\frac{2}{2}\right) + 3 - d &= 0 \\
\frac{27}{8} - \frac{90}{8} + \frac{24}{8} - d &= 0 \\
\frac{-39}{8} &= d.
\end{aligned}$$

(d) You are given that $(x-2)$ and $(x+1)$ are factors of the polynomial $f(x) = x^4 - 8x^3 + hx^2 + kx + 6$. Find h and k.

Solution: We need both $f(2)$ and $f(-1)$ to equal zero.

$$\begin{aligned}
0 = f(2) &= (2)^4 - 8(2)^3 + h(2)^2 + k(2) + 6 \\
&= 16 - 64 + 4h + 2k + 6 \\
&= 4h + 2k - 42. \\
0 = f(-1) &= (-1)^4 - 8(-1)^3 + h(-1)^2 + k(-1) + 6 \\
&= 1 + 8 + h - k + 6 \\
&= h - k + 15.
\end{aligned}$$

Therefore from the second equation, $h = k - 15$, and plugging into the first one, we get

$$4(k - 15) + 2k - 42 = 0$$

$$
\begin{aligned}
4k - 60 + 2k - 42 &= 0 \\
6k - 102 &= 0 \\
6k &= 102 \\
k &= 17 \\
h &= 17 - 15 = 2.
\end{aligned}
$$

Therefore the only solution is $k = 17$, $h = 2$.

7. You are given that $2 + i$ is a zero of the polynomial $p(x) = x^4 - 4x^3 + 9x^2 - 16x + 20$. Write $p(x)$ as a product of linear factors. What are the roots of the equation $p(x) = 0$?

Solution: Since $2 + i$ is a zero, so is $2 - i$, and therefore:

$$
\begin{aligned}
(x - (2 + i))(x - (2 - i)) &= (x - 2 - i)(x - 2 + i) \\
&= x^2 - 2x + ix - 2x + 4 - 2i - ix + 2i - i^2 \\
&= x^2 - 4x + 4 - (-1) \\
&= x^2 - 4x + 5
\end{aligned}
$$

is a factor of $p(x)$.

$$
\begin{array}{r|l}
 & x^2 + 4 \\
\hline
x^2 - 4x + 5 & x^4 - 4x^3 + 9x^2 - 16x + 20 \\
 & \underline{x^4 - 4x^3 + 5x^2} \\
 & 4x^2 - 16x + 20 \\
 & 4x^2 - 16x + 20
\end{array}
$$

Therefore,

$$x^4 - 4x^3 + 9x^2 - 16x + 20 = (x^2 - 4x + 5)(x^2 + 4)$$

and since $x^2 + 4 = x^2 - (2i)^2 = (x - 2i)(x + 2i)$, we have that we can write $p(x)$ as:

$$p(x) = (x - (2 + i))(x - (2 - i))(x - 2i)(x + 2i).$$

Therefore the roots of $p(x) = 0$ are

$$2 + i, 2 - i, 2i, -2i.$$

8. In each case your response should refer by number to appropriate results in the textbook as needed.

   (a) If a polynomial of degree n with real coefficients does not have n real zeros (counting multiplicity) then it must have an irreducible quadratic factor. Justify this statement.

Solution: Let $f(x)$ be a polynomial of degree n with real coefficients that does not have n real zeros (counting multiplicity). The polynomial has n zeros by Theorem 2.4 (or FTA (II)). Since it does not have n real zeros, it has at least one zero, say λ, that is not real. Then by Theorem 2.5, $\overline{\lambda}$ is also a zero. Then by Theorem 2.2, $(x-\lambda)(x-\overline{\lambda})$ is an irreducible quadratic factor (since it's zeros are not real).

(b) If r is a zero of a polynomial $f(x)$ of multiplicity 5 and a zero of the polynomial $g(x)$ of multiplicity 7, must it also be a zero of the polynomial $h(x) = f(x) + g(x)$? If so, can we determine its multiplicity? If so, what is it? If not, why not?

Solution: Assume r is a zero of a polynomial $f(x)$ of multiplicity 5 and a zero of the polynomial $g(x)$ of multiplicity 7. Then by Theorem 2.4 (or Theorem 2.2), and the definition of "multiplicity", we can write

$$f(x) = (x-r)^5 p(x) \qquad \text{and} \qquad g(x) = (x-r)^7 q(x)$$

where $p(r) \neq 0$ and $q(r) \neq 0$. Thus

$$\begin{aligned} h(x) &= f(x) + g(x) \\ &= (x-r)^5 p(x) + (x-r)^7 q(x) \\ &= (x-r)^5 (p(x) + (x-r)^2 q(x)). \end{aligned}$$

Thus $h(r) = 0$ (and so r is a root of $h(x) = 0$), and the multiplicity is 5, since we also know that $p(r) + (r-r)^2 q(r) = p(r) \neq 0$. Therefore r is a zero of $h(x)$ and the multiplicity can be determined to be 5.