Attempt all questions and show all your work. Due November 13, 2015.

1. Prove using mathematical induction that for any $n \geq 2$, and collection of n $m \times m$ matrices $A_1, A_2, \ldots, A_n$,

$$\det(A_1 A_2 \cdots A_n) = \det(A_1)\det(A_2)\cdots\det(A_n).$$

2. Prove using mathematical induction that for any $n \geq 1$, the determinant of an upper-triangular $n \times n$ matrix is the product of its diagonal entries.

3. Is it true that for any two matrices A and B,

$$\det(A+B) = \det(A) + \det(B)?$$

   If so, prove it. If not, find a counter example.

4. Solve the following system using Cramer's Rule:

$$\begin{array}{rcrcrcr} x_1 & & & + & 3x_3 & = & -1 \\ & - & x_2 & + & 2x_3 & = & -9 \\ 2x_1 & + & x_2 & & & = & 15 \end{array}$$

5. Prove the following property: for all $a, b, c \in \mathbb{R}$, $a \neq 0$, $b \neq 0$, $c \neq 0$,

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

6. (a) Let $c \in \mathbb{R}$. Prove using mathematical induction that for any $n \geq 1$ and any $n \times n$ matrix A, $|cA| = c^n|A|$.

   (b) A square matrix is called **skew-symmetric** if $A^T = -A$. Use part (a) and a property of determinants when taking transposes to show that every skew-symmetric 1001×1001 matrix has determinant 0.

7. An **elementary matrix** is a matrix which is one elementary row operation away from the identity matrix. For instance,

$$E_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

   are all elementary matrices.

   (a) Let k be any real number, $k \neq 0$. Find an elementary matrix with determinant k.

   (b) **BONUS: 3 MARKS**. Let E be an $n \times n$ elementary matrix formed by performing row operation r to the identity I_n. Let A be any $n \times n$ matrix. Then the matrix product EA will result in the result of performing r to A. Use this fact, and properties of determinants to formally prove the following theorem: If A is an $n \times n$ matrix such that the row reduced row echelon form of A is I_n, then $\det(A) \neq 0$.

8. Let $\mathbf{u} = [1, 1, 1]$, $\mathbf{v} = [-1, 2, 5]$, $\mathbf{w} = [0, 1, 1]$. Calculate each of the following:
   (a) $(2\mathbf{u} + \mathbf{v}) \bullet (\mathbf{v} - 3\mathbf{w})$
   (b) $||\mathbf{u}|| - 2||\mathbf{v}|| + ||(-3)\mathbf{w}||$

9. Prove the associative rule for addition of vectors in E^3

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

   in the following two different ways:
   (a) by writing each of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in terms of their coordinates and simplifying both sides algebraically in coordinate form
   (b) by a geometric argument using arrow representations for $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$

10. Find the points where the plane $3x - 2y + 5z = 30$ meets each of the x, y and z axes in E^3. Use these "intercepts" to provide a neat sketch of the plane.

11. (a) Find an equation for the line through points $(1, 3)$ and $(5, 4)$ in parametric form.
    (b) Find an equation for the line through points $(1, 2, 3)$ and $(5, 5, 0)$ in parametric form.