MATH 1210 Assignment 4 Solutions 16R-T1

Attempt all questions and show all your work. Due November 13, 2015.

1. Prove using mathematical induction that for any $n \geq 2$, and collection of n $m \times m$ matrices $A_1, A_2, \ldots, A_n$,

$$\det(A_1 A_2 \cdots A_n) = \det(A_1)\det(A_2)\cdots\det(A_n).$$

Solution: Fix $m \geq 1$. For all $n \geq 2$, let P_n denote the statement that for any collection of n $m \times m$ matrices $A_1, A_2, \ldots, A_n$,

$$\det(A_1 A_2 \cdots A_n) = \det(A_1)\det(A_2)\cdots\det(A_n).$$

Base Case. The statement P_2 says that for any collection of 2 $m \times m$ matrices A, B,

$$\det(AB) = \det(A)\det(B).$$

This is true by Theorem 7.6 in the text.

Inductive Step. Fix $k \geq 2$ and suppose that P_k holds, that is, for any collection of k $m \times m$ matrices $A_1, A_2, \ldots, A_k$,

$$\det(A_1 A_2 \cdots A_k) = \det(A_1)\det(A_2)\cdots\det(A_k).$$

It remains to show that P_{k+1} holds, that is, for any collection of $k+1$ $m \times m$ matrices $A_1, A_2, \ldots, A_{k+1}$,

$$\det(A_1 A_2 \cdots A_{k+1}) = \det(A_1)\det(A_2)\cdots\det(A_{k+1}).$$

Let $A_1, A_2, \ldots, A_{k+1}$ be $m \times m$ matrices. Then

$$\begin{aligned}
\det(A_1 A_2 \cdots A_{k+1}) &= \det(\ (A_1 A_2 \cdots A_k)\ A_{k+1}\ ) \\
&= \det(A_1 A_2 \cdots A_k)\ \det(A_{k+1}) && \text{(by } P_2\text{)} \\
&= \det(A_1)\det(A_2)\cdots\det(A_k)\det(A_{k+1}) && \text{(by } P_k\text{).}
\end{aligned}$$

Therefore P_{k+1} holds. Thus by PMI, for all $n \geq 2$, P_n holds.

2. Prove using mathematical induction that for any $n \geq 1$, the determinant of an upper-triangular $n \times n$ matrix is the product of its diagonal entries.

Solution: For any $n \geq 1$, let P_n denote the statement that the determinant of every upper-triangular $n \times n$ matrix is the product of its diagonal entries.

Base Case. The statement P_1 says that the determinant of every upper-triangular 1×1 matrix is the product of its diagonal entries. Every 1×1 matrix $A = [a_{1,1}]$

is upper-triangular, and $|A| = a_{1,1}$, which is the product of the diagonal entries. Therefore P_1 holds.

Inductive Step. Fix $k \geq 1$ and assume that P_k holds, that is, the determinant of every upper-triangular $k \times k$ matrix is the product of its diagonal entries. It remains to show that P_{k+1} holds, that is, the determinant of every upper-triangular $k+1 \times k+1$ matrix is the product of its diagonal entries.

Let $A = [a_{i,j}]_{k+1 \times k+1}$ be an upper-triangular matrix. First some notation: let $A_{i,j}$ denote the matrix formed from A by removing row i and column j.

The last row of A is all zeros, except for the last entry, $a_{k+1,k+1}$. Therefore expanding across the bottom row, we have:

$$\begin{aligned} |A| &= \sum_{j=1}^{k+1} a_{k+1,j} C_{k+1,j} \\ &= a_{k+1,k+1} C_{k+1,k+1} \\ &= a_{k+1,k+1} (-1)^{k+1+k+1} |A_{k+1,k+1}| \\ &= a_{k+1,k+1} (-1)^{2k+2} |A_{k+1,k+1}| \\ &= a_{k+1,k+1} |A_{k+1,k+1}|. \end{aligned}$$

Note that $A_{k+1,k+1}$ is a $k \times k$ upper-triangular matrix. Therefore by P_k, the determinant of $A_{k+1,k+1}$ is the product of the diagonal entries, that is,

$$|A_{k+1,k+1}| = a_{1,1} a_{2,2} \cdots a_{k,k}.$$

Put these together and we get

$$|A| = a_{k+1,k+1} |A_{k+1,k+1}| = a_{k+1,k+1} a_{1,1} a_{2,2} \cdots a_{k,k}.$$

Therefore P_{k+1} holds. Thus by PMI, for all $n \geq 1$, P_n holds.

3. Is it true that for any two matrices A and B,

$$\det(A + B) = \det(A) + \det(B)?$$

If so, prove it. If not, find a counter example.

Solution: No. For instance,

$$\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, \qquad \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0,$$

but

$$\left| \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}.$$

4. Solve the following system using Cramer's Rule:

$$\begin{array}{rcrcrcr} x_1 & & & + & 3x_3 & = & -1 \\ & - & x_2 & + & 2x_3 & = & -9 \\ 2x_1 & + & x_2 & & & = & 15 \end{array}$$

Solution:

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -9 \\ 15 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -1 & 0 & 3 \\ -9 & -1 & 2 \\ 15 & 1 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & -1 & 3 \\ 0 & -9 & 2 \\ 2 & 15 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -9 \\ 2 & 1 & 15 \end{bmatrix}$$

Then $|A| = 4$, $|A_1| = 20$, $|A_2| = 20$, and $|A_3| = -8$. Therefore, by Cramer's rule:

$$x_1 = \frac{|A_1|}{|A|} = \frac{20}{4} = 5$$
$$x_2 = \frac{|A_2|}{|A|} = \frac{20}{4} = 5$$
$$x_3 = \frac{|A_3|}{|A|} = \frac{-8}{4} = -2.$$

5. Prove the following property: for all $a, b, c \in \mathbb{R}$, $a \neq 0$, $b \neq 0$, $c \neq 0$,

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Solution:

$$\begin{aligned} & \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} R_2 \leftarrow R_2 - R_1 \\ = & \begin{vmatrix} 1+a & 1 & 1 \\ -a & b & 0 \\ 1 & 1 & 1+c \end{vmatrix} R_3 \leftarrow R_3 - (1+c)R_1 \\ = & \begin{vmatrix} 1+a & 1 & 1 \\ -a & b & 0 \\ 1-(1+a)(1+c) & -c & 0 \end{vmatrix} \end{aligned}$$

$$= \begin{vmatrix} 1+a & 1 & 1 \\ -a & b & 0 \\ -a-c-ac & -c & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -a & b \\ -a-c-ac & -c \end{vmatrix}$$

$$= ac - b(-a-c-ac) = ac + ba + bc + abc = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

6. (a) Let $c \in \mathbb{R}$. Prove using mathematical induction that for any $n \geq 1$ and any $n \times n$ matrix A, $|cA| = c^n|A|$.

Solution: Fix $c \in \mathbb{R}$. For all $n \geq 1$, let P_n denote the statement that for any $n \times n$ matrix A, $|cA| = c^n|A|$.

Base Case. The statement P_1 says that for any 1×1 matrix $A = [a_{1,1}]$, $|cA| = c|A|$.

$$|cA| = |[ca_{1,1}]| = ca_{1,1} = c|[a_{1,1}]| = c|A|.$$

Therefore P_1 holds.

Inductive Step. Fix $k \geq 1$ and assume that P_k holds, that is, for any $k \times k$ matrix A, $|cA| = c^k|A|$. It remains to show that P_{k+1} holds, that is, for any $k+1 \times k+1$ matrix A, $|cA| = c^{k+1}|A|$.

First some notation: let $A_{i,j}$ denote the matrix formed from A by removing row i and column j. Then expanding across the first row we have:

$$\begin{aligned}
|cA| &= \sum_{j=1}^{k+1} ca_{1,j}C_{1,j} \\
&= \sum_{j=1}^{k+1} ca_{1,j}(-1)^{1+j}|(cA)_{1,j}| \\
&= \sum_{j=1}^{k+1} ca_{1,j}(-1)^{1+j}c^k|A_{1,j}| \qquad \text{By } P_k \text{ since } (cA)_{1,j} \text{ is a } k \times k \text{ matrix} \\
&= \sum_{j=1}^{k+1} ca_{1,j}(-1)^{1+j}c^k|A_{1,j}| \qquad \text{By } P_k \text{ since } (cA)_{1,j} \text{ is a } k \times k \text{ matrix} \\
&= c^{k+1}\sum_{j=1}^{k+1} a_{1,j}(-1)^{1+j}|A_{1,j}| \\
&= c^{k+1}|A|.
\end{aligned}$$

Therefore P_{k+1} holds, and thus by PMI, for all $n \geq 1$, P_n holds.

(b) A square matrix is called **skew-symmetric** if $A^T = -A$. Use part (a) and a property of determinants when taking transposes to show that every skew-symmetric

1001×1001 matrix has determinant 0.

Solution: Let A be a skew-symmetric matrix. Then $A^T = -A$. Taking the determinant of both sides, we get

$$\begin{aligned} |A^T| &= |A| \\ |-A| &= |(-1)A| \\ &= (-1)^{1001}|A| \\ &= -|A|. \end{aligned}$$

Thus $|A| = -|A|$, and so $2|A| = 0$, thus $|A| = 0$.

7. An **elementary matrix** is a matrix which is one elementary row operation away from the identity matrix. For instance,

$$E_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

are all elementary matrices.

(a) Let k be any real number, $k \neq 0$. Find an elementary matrix with determinant k.

Solution: For any $k \neq 0$, $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ is an elementary matrix (for the row operation $R_1 \leftarrow kR_1$), and has determinant k.

(b) **BONUS: 3 MARKS**. Let E be an $n \times n$ elementary matrix formed by performing row operation r to the identity I_n. Let A be any $n \times n$ matrix. Then the matrix product EA will result in the result of performing r to A. Use this fact, and properties of determinants to formally prove the following theorem: If A is an $n \times n$ matrix such that the row reduced row echelon form of A is I_n, then $\det(A) \neq 0$.

Solution: Assume A is an $n \times n$ matrix with RREF I_n. Then there exist row operations $r_1, r_2, \ldots, r_k$ such that if we perform them to A (starting with r_1 and proceeding in order), we get I_n. That is, there exist elementary matrices $E_1, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = I_n$. Therefore $\det(E_k \cdots E_2 E_1 A) = \det(I_n) = 1$. But we know from question 1 above that $\det(E_k \cdots E_2 E_1 A) = \det(E_k) \cdots \det(E_1) \det(A)$. Therefore $\det(A)$ cannot equal zero (since if it did, the left hand side would be zero, not 1).

8. Let $\mathbf{u} = [1, 1, 1]$, $\mathbf{v} = [-1, 2, 5]$, $\mathbf{w} = [0, 1, 1]$. Calculate each of the following:

(a) $(2\mathbf{u} + \mathbf{v}) \bullet (\mathbf{v} - 3\mathbf{w})$

Solution:

$$(2\mathbf{u} + \mathbf{v}) \bullet (\mathbf{v} - 3\mathbf{w}) = (2[1, 1, 1] + [-1, 2, 5]) \bullet ([-1, 2, 5] - 3[0, 1, 1])$$

$$
\begin{aligned}
&= ([2,2,2]+[-1,2,5]) \bullet ([-1,2,5]-[0,3,3]) \\
&= [1,4,7] \bullet [-1,-1,2] \\
&= -1-4+14 = 9.
\end{aligned}
$$

(b) $||\mathbf{u}|| - 2||\mathbf{v}|| + ||(-3)\mathbf{w}||$

Solution:

$$
\begin{aligned}
||\mathbf{u}|| - 2||\mathbf{v}|| + ||(-3)\mathbf{w}|| &= ||[1,1,1]|| - 2||[-1,2,5]|| + ||(-3)[0,1,1]|| \\
&= \sqrt{1^2+1^2+1^2} - 2\sqrt{1^2+2^2+5^2} + ||[0,-3,-3]|| \\
&= \sqrt{3} - 2\sqrt{30} + \sqrt{3^2+3^2} \\
&= \sqrt{3} - 2\sqrt{30} + \sqrt{18}.
\end{aligned}
$$

9. Prove the associative rule for addition of vectors in E^3

$$(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$$

in the following two different ways:

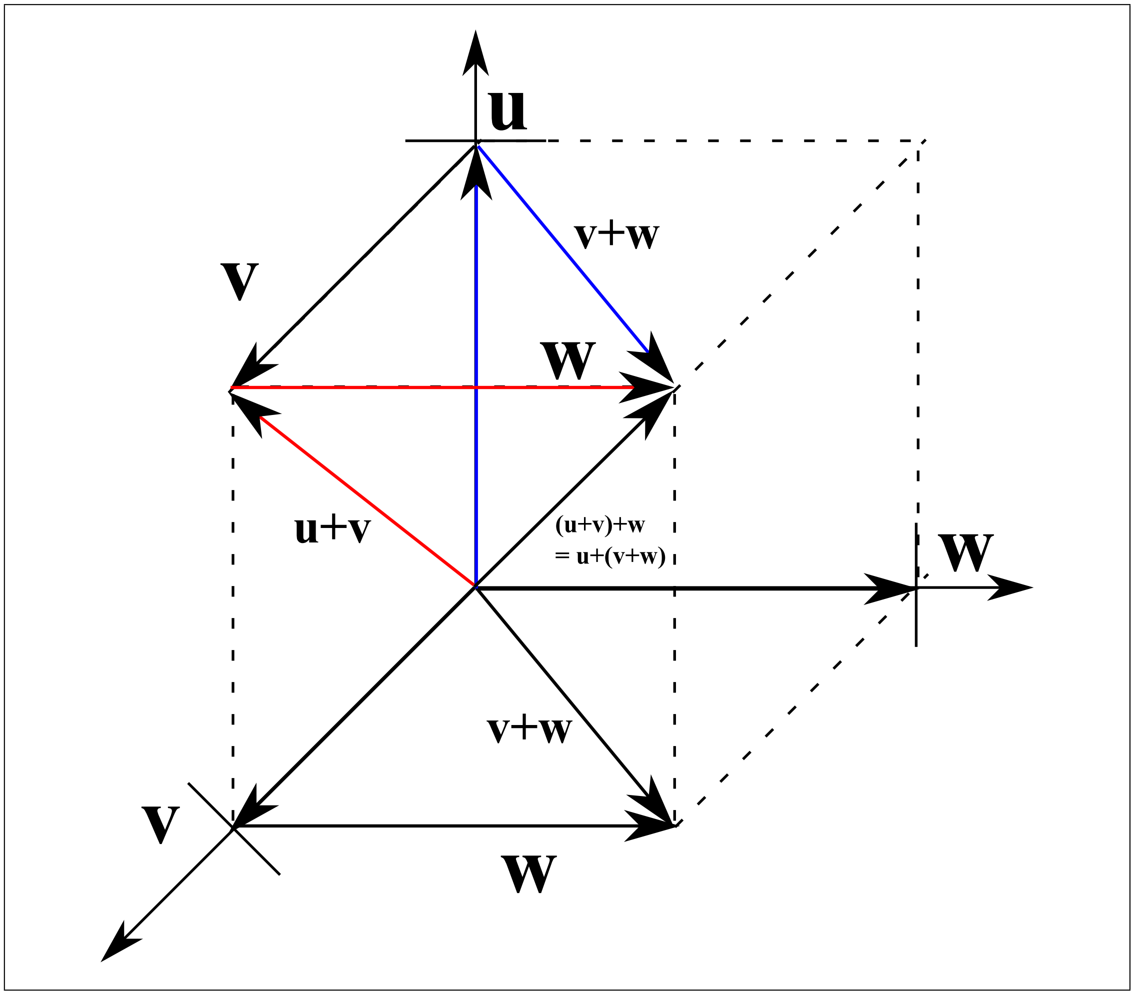
(a) by writing each of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in terms of their coordinates and simplifying both sides algebraically in coordinate form

Solution: Let $\mathbf{u} = (a,b,c)$, $\mathbf{v} = (d,e,f)$, $\mathbf{w} = (x,y,z)$. Then

$$
\begin{aligned}
(\mathbf{u}+\mathbf{v})+\mathbf{w} &= ((a,b,c)+(d,e,f)) + (x,y,z) \\
&= (a+d, b+e, c+f) + (x,y,z) \\
&= ((a+d)+x, (b+e)+y, (c+f)+z) \\
&= (a+(d+x), b+(e+y), c+(f+z)) \\
&= (a,b,c) + (d+x, e+y, f+z) \\
&= (a,b,c) + ((d,e,f)+(x,y,z)) \\
&= \mathbf{u}+(\mathbf{v}+\mathbf{w}).
\end{aligned}
$$

(b) by a geometric argument using arrow representations for $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$

Solution:



10. Find the points where the plane $3x - 2y + 5z = 30$ meets each of the x, y and z axes in E^3. Use these "intercepts" to provide a neat sketch of the plane.

Solution: Here are the formulae for the axes:

$$x\text{-axis:} \qquad \mathbf{x} = (t, 0, 0),\ t \in \mathbb{R}$$

$$y\text{-axis:} \qquad \mathbf{x} = (0, t, 0),\ t \in \mathbb{R}$$

$$z\text{-axis:} \qquad \mathbf{x} = (0, 0, t),\ t \in \mathbb{R}$$

So to find the intersection of this plane and the x-axis, just plug in $y = z = 0$:

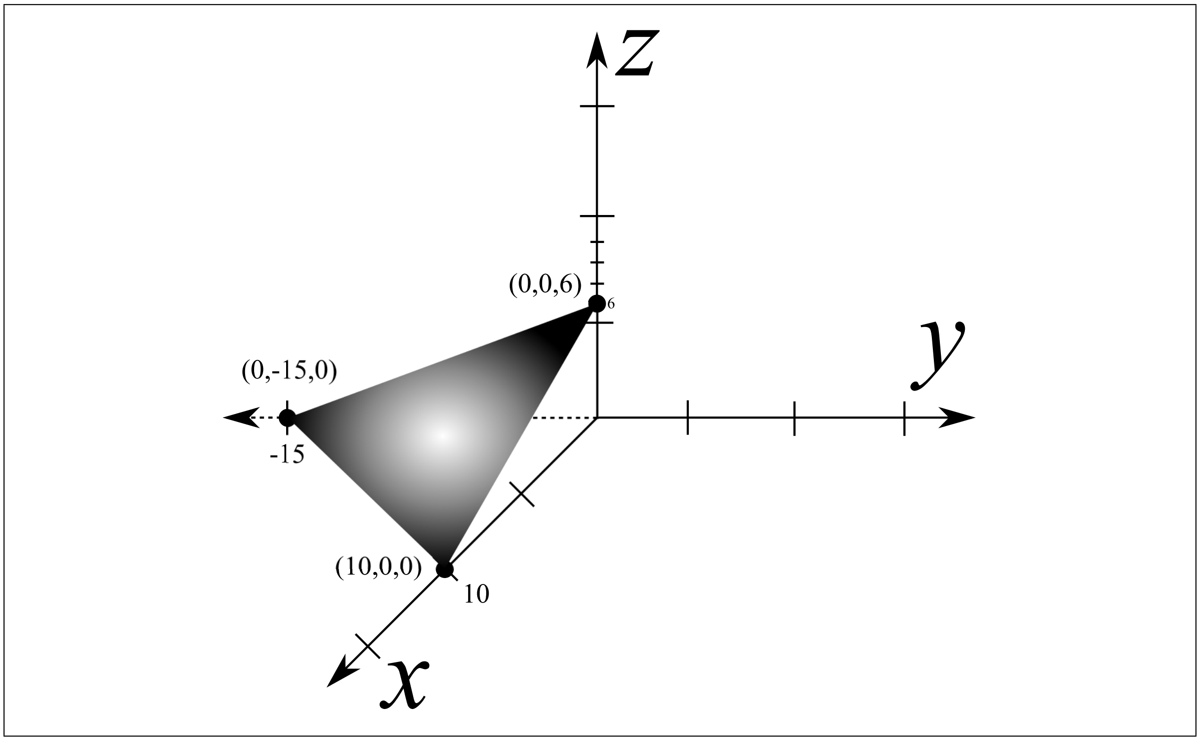
$$3x = 30 \implies x = 10 \implies (10, 0, 0)$$

Similarly,

$$-2y = 30 \implies y = -15 \implies (0, -15, 0)$$

$$5z = 30 \implies z = 6 \implies (0, 0, 6)$$

Therefore we have the following plane:



11. (a) Find an equation for the line through points $(1, 3)$ and $(5, 4)$ in parametric form.

Solution: The vector $\mathbf{v} = (5, 4) - (1, 3) = (4, 1)$ is along the line. Therefore the line in point-parallel form is:

$$\mathbf{x} = (1, 3) + t(4, 1),\ t \in \mathbb{R}$$

which in parametric form becomes

$$x = 1 + 4t, \qquad y = 3 + t,\ t \in \mathbb{R}.$$

(b) Find an equation for the line through points $(1, 2, 3)$ and $(5, 5, 0)$ in parametric form.

Solution: The vector $\mathbf{v} = (5, 5, 0) - (1, 2, 3) = (4, 3, -3)$ is along the line. Therefore the line in point-parallel form is:

$$\mathbf{x} = (5, 5, 0) + t(4, 3, -3),\ t \in \mathbb{R}$$

which in parametric form becomes

$$x = 5 + 4t, \qquad y = 5 + 3t, \qquad z = -3t,\ t \in \mathbb{R}.$$