Attempt all questions and show all your work. Due December 7, 2015.

1. Let $Q(1,3,-2)$ be a point in xyz-space, $P_1 : x - y + 2z = 1$ and $P_2 : 2x + y + z = 4$ be two planes in xyz-space.

   (a) Find an equation of the plane through Q that is parallel to P_1.

   Solution: If the planes are parallel, then they have the same normal vector, so the normal to the plane we seek is $(1,-1,2)$ and the plane equation takes the form $x - y + 2z = d$, where the value of d is such that Q belongs to the plane. Substituting the coordinates of Q, i.e., $(1,3,-2)$, for (x,y,z), we have

   $$(1) - (3) + 2(-2) = d \Leftrightarrow d = -6$$

   and thus the equation we seek is $x - y + 2z = -6$.

   (b) Find an equation of the plane through Q and perpendicular to the line of the intersection of P_1 and P_2.

   Solution: First, we seek the line of intersection of P_1 and P_2:

   $$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 4 \end{array}\right] \overset{R_2 \leftarrow R_2 - 2R_1}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 2 \end{array}\right]$$
   $$\overset{R_2 \leftarrow R_2/3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2/3 \end{array}\right] \overset{R_1 \leftarrow R_1 + R_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5/3 \\ 0 & 1 & -1 & 2/3 \end{array}\right].$$

   Thus, for $z = t \in \mathbb{R}$, $x = 5/3 - t$ and $y = 2/3 + t$, i.e., $(x,y,z) = (-1,1,1)t + (5/3, 2/3, 0)$. The plane we seek is perpendicular to this line, i.e., has the direction of this line, $(-1,1,1)$, as its normal vector. We can then proceed using the point normal form or, as we will, as in part (a): the plane we seek has equation $-x + y + z = d$ and contains the point Q, so substituting the coordinates of Q gives us d:

   $$-(1) + (3) + (-2) = d \Leftrightarrow d = 0.$$

   Thus the plane has equation $-x + y + z = 0$.

2. Let $P_1 = (1,-1,2)$, $P_2 = (2,0,4)$ and $P_3 = (4,0,3)$ be three points in xyz-space.

   (a) Is the triangle determined by those points a right triangle? Explain your answer.

   Solution: If the triangle is a right triangle, then two of $\overrightarrow{P_1P_2}$, $\overrightarrow{P_1P_3}$ and $\overrightarrow{P_2P_3}$ must be perpendicular. We know that for two nonzero vectors a, b, $a \perp b \Leftrightarrow a \cdot b = 0$. We have

   $$\overrightarrow{P_1P_2} = (1,1,2), \quad \overrightarrow{P_1P_3} = (3,1,1) \quad \text{and} \quad \overrightarrow{P_2P_3} = (2,0,-1),$$

   so

   $$\overrightarrow{P_1P_2} \cdot \overrightarrow{P_1P_3} = 3+1+2 = 6, \quad \overrightarrow{P_1P_2} \cdot \overrightarrow{P_2P_3} = 2-2 = 0 \quad \text{and} \quad \overrightarrow{P_1P_3} \cdot \overrightarrow{P_2P_3} = 6-1 = 5.$$

Thus $\overrightarrow{P_1P_2} \perp \overrightarrow{P_1P_3}$ and the triangle $P_1P_2P_3$ is a right triangle with right angle at P_3.

(b) Find an equation of the plane containing P_1, P_2 and P_3.

Solution: The normal to the plane we seek is perpendicular to the cross product of any pair of the vectors $\overrightarrow{P_1P_2}$, $\overrightarrow{P_1P_3}$ and $\overrightarrow{P_2P_3}$. We take for instance $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_3}$ [the latter has a zero coordinate that will make computations easier]:

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ 2 & 0 & -1 \end{vmatrix} = (-1, 5, -2).$$

Thus the plane has equation $-x + 5y - 2z = d$ or, to get rid of some minus signs, $x - 5y + 2z = e$, where we seek the value of e by noting that the plane equation must hold for any of P_1, P_2 or P_3. Take P_2, for instance:

$$(2) - 5(0) + 2(4) = e \Leftrightarrow e = 6,$$

so the equation of the plane is $x - 5y + 2z = 6$.

3. Find the inverse of the matrix or explain why the inverse does not exists.

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Solution: Compute $\det(A)$ by expanding along the bottom row:

$$\det(A) = \begin{vmatrix} 2 & 3 \\ 5 & 3 \end{vmatrix} + 8 \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 6 - 15 + 40 - 32 = -1,$$

so A is invertible. We compute the inverse using the adjugate. The matrix of cofactors is

$$C = \begin{bmatrix} 40 & -13 & -5 \\ -16 & 5 & 2 \\ -9 & 3 & 1 \end{bmatrix},$$

so

$$A^{-1} = \frac{1}{\det(A)} C^T = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}.$$

(b) $B = \begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$

Solution: We have

$$\det(B) = \begin{vmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix} \underset{R_3 \leftarrow R_3 - 4R_1}{\overset{R_2 \leftarrow R_2 + 2R_1}{=}} \begin{vmatrix} -1 & 3 & -4 \\ 0 & 10 & -7 \\ 0 & -10 & 7 \end{vmatrix}$$

and thus $\det(B) = 0$ since the third row is a multiple of the second (and thus adding, for instance, row 2 to row 3, would lead to a row of zeros). So B is not invertible.

4. Find all values of c, if any, for which the matrix $A = \begin{bmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{bmatrix}$ is invertible.

Solution: A is invertible iff its determinant is nonzero. Expanding along, say, the first row, we find

$$\det(A) = c \begin{vmatrix} c & 1 \\ 1 & c \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & c \end{vmatrix} = c(c^2 - 1) - c = c(c^2 - 2).$$

Therefore, A is invertible iff $c \neq 0$ and $c^2 - 2 \neq 0$, i.e., $c \neq 0$, $c \neq \sqrt{2}$ and $c \neq -\sqrt{2}$.

5. Show that if A is invertible, then $\det(A^{-1}) = \det(A)^{-1}$. Deduce a formula for the determinant of $4A^{-1}$, when A is an invertible $n \times n$-matrix.

Solution: If A is invertible, then A^{-1} exists such that $AA^{-1} = I$. Take the determinant of both sides and use the fact that $\det(AB) = \det(A)\det(B)$:

$$\det(AA^{-1}) = \det(I) \Leftrightarrow \det(A)\det(A^{-1}) = \det(I) \Leftrightarrow \det(A)\det(A^{-1}) = 1$$

and thus, dividing both sides of the latter equality by $\det(A)$,

$$\det(A^{-1}) = \det(A)^{-1}.$$

Recall that if all entries in a row of A are multiplied by some constant k, then the determinant of A is multiplied by k. So, if A is $n \times n$, if all entries of A are multiplied by k, then the determinant of A is multiplied by k^n. Therefore,

$$\det(4A^{-1}) = 4^n \det(A^{-1}) = \frac{4^n}{\det(A)}.$$

6. Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 4 & 1 & 2 \\ 0 & 3 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 2 & 1 & 4 & 1 & 0 \\ -1 & 3 & 4 & 1 & 6 \end{pmatrix}.$$

Evaluate each of the following:

(a) The (2,3) cofactor of A.

Solution: $c_{23} = -m_{23} = -3$.

(b) The 3rd row of $B^T A$.

Solution: The third row of $B^T A$ comes from multiplying the third row of B^T by the successive columns of A. We have

$$\begin{pmatrix} 0 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 4 & 1 & 2 \\ 0 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 16 & 16 & 4 \end{pmatrix}.$$

(c) $\det(2(A^{-1})^T)$.

Solution: We have $\det(2(A^{-1})^T) = 2^3 \det((A^{-1})^T) = 8 \det(A^{-1}) = 8/\det(A)$ and

$$\det(A) = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 4 & 1 \\ 0 & 3 \end{vmatrix} = -7 + 12 = 5.$$

Thus $\det(2(A^{-1})^T) = 8/5$.

7. Writing the system

$$\begin{array}{llll} x_1 & & +x_3 & = 4 \\ 2x_1 & +3x_2 & +5x_3 & = -3 \\ x_1 & & +2x_3 & = 0 \end{array}$$

as $A\mathbf{x} = \mathbf{b}$,

(a) find the inverse matrix A^{-1};

Solution: The matrix takes the form

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}.$$

We compute the determinant by expanding along the second column:

$$\det(A) = 3 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 3,$$

so the matrix A is invertible. We invert it, for example, by row reduction. We have

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \underset{R_3 \leftarrow R_3 - R_1}{\overset{R_2 \leftarrow R_2 - 2R_1}{\longrightarrow}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\underset{R_2 \leftarrow R_2 - 3R_1}{\overset{R_1 \leftarrow R_1 - R_3}{\longrightarrow}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 3 & 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \overset{R_2 \leftarrow R_2/3}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] = [I|A^{-1}].$$

(b) find the solution to the system $A^T\mathbf{x} = \mathbf{b}$ by using (a).

Solution: Since $\det(A) = \det(A^T)$, the system $A^T\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = (A^T)^{-1}\mathbf{b}$. Recall that $(A^T)^{-1} = (A^{-1})^T$. [Indeed, suppose A is invertible. Then $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$, thus the inverse of A^T is $(A^{-1})^T$.]

Thus

$$\mathbf{x} = (A^T)^{-1}\mathbf{b} = (A^{-1})^T\mathbf{b} = \begin{bmatrix} 2 & 0 & -1 \\ \frac{1}{3} & \frac{1}{3} & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ \frac{1}{3} \\ -4 \end{bmatrix}.$$

8. Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}.$$

(a) Calculate $\det(A)$. Is A invertible? Explain.

Solution: Expanding along the top row, we get

$$\det(A) = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1,$$

so A is invertible since its determinant is nonzero.

(b) If A is invertible, find A^{-1} by using the row reduction method.

Solution: We have

$$[A|I] = \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_2 \leftrightarrow R_3}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \overset{R_2 \leftarrow R_2 - R_1}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\overset{R_2 \leftarrow R_2 - R_3}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \overset{R_1 \leftarrow R_1 - R_2}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$= [I|A^{-1}],$$

i.e.,

$$A^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

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9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that reflects any vector about the xz-plane then multiplies its length by $h \neq 0$. Find the value of h so that the vector $(2, 4, 3)$ is the image through T of the vector $\left(-\frac{2}{3}, \frac{4}{3}, -1\right)$.

Solution: First, the reflection about the xz-plane acts on $\vec{\imath}, \vec{\jmath}, \vec{k}$ as follows: $T(\vec{\imath}) = \vec{\imath}$, $T(\vec{\jmath}) = -\vec{\jmath}$ and $T(\vec{k}) = \vec{k}$. Multiplying the length of the image by h then results in the following action: $T(\vec{\imath}) = h\vec{\imath}$, $T(\vec{\jmath}) = -h\vec{\jmath}$ and $T(\vec{k}) = h\vec{k}$. As a consequence, the matrix A associated to the linear transformation takes the form

$$A = \begin{bmatrix} h & 0 & 0 \\ 0 & -h & 0 \\ 0 & 0 & h \end{bmatrix}.$$

We are told that

$$(2, 4, 3) = T(-2/3, 4/3, -1),$$

or, in other words,

$$A \begin{bmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ -1 \end{bmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}.$$

Thus

$$\begin{aligned} -\frac{2}{3}h &= 2 \\ -\frac{4}{3}h &= 4 \\ -h &= 3, \end{aligned}$$

so clearly, $h = -3$. [Which means that there is an additional reflection about the origin.]