UNIVERSITY OF MANITOBA

DATE: October 20, 2015

MIDTERM EXAMINATION SOLUTIONS

TITLE PAGE

DEPARTMENT & COURSE NO: MATH 1210

TIME: 60 minutes

EXAMINATION: Techniques of Classical and Linear Algebra

EXAMINER: Borgersen/Arino

Unique Identifier Sticker:

DO NOT WRITE ABOVE THIS LINE. ______

INSTRUCTIONS TO STUDENTS:

This is a 60 minute exam. **Please show your work clearly.**

No texts or notes are permitted. No calculators are permitted. Cell phones, electronic translators, and other electronic devices are **not** permitted.

This exam has a title page and 8 pages of questions, including 2 blank pages for rough/extra work. Please check that you have all the pages.

The value of each question is indicated beside the statement of the question. The total value of all questions is 73 points.

If you need more scrap paper, use the back of the question pages. **Anything written on the back of a page will not be marked.** If you need more space to answer a question (that you want marked), write it on one of the scrap pages at the back.

INDICATE YOUR SECTION:

- ❑ A01 (9:30-10:20 MWF) R. Borgersen
- ❑ A02 (13:30–14:20 MWF) J. Arino
- ❑ A03 (13:30–14:20 MWF) R. Borgersen

FIRST NAME: ______

LAST NAME: ______

STUDENT NUMBER: ______

PAPER NUMBER: ______
(Paper Number is found at the top)

SIGNATURE: (in ink)

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(I understand that cheating is a serious offense)

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1. [15 points] Prove by induction that for all $n \geq 1$, $\sum_{i=1}^{2n} 3i - 2 = n(6n-1)$. Show all your work, use complete sentences, and use the style done in class.

Solution: For any integer $n \geq 1$, let P_n denote the statement, $1+4+7+10+\ldots+(6n-2) = n(6n-1)$ which is an equivalent equation to, $\sum_{i=1}^{2n} 3i-2 = n(6n-1)$.

Base Case. The statement P_1 says that, $1+4=5$ and $1(6(1)-1)=5$ are equivalent, which is true.

Inductive Step. Fix $k \geq 1$ and suppose that P_k holds, that is, $1+4+7+\ldots+(6k-2) = k(6k-1)$.

It remains to show that P_{k+1} holds, that is, $1+4+7+\ldots+(6(k+1)-2) = (k+1)(6(k+1)-1)$.

$$\begin{aligned}
1+4+7+\ldots+(6(k+1)-2) &= 1+4+7+\ldots+(6k+4) \\
&= 1+4+7+\ldots+(6k-2)+(6k+1)+(6k+4) \\
&= k(6k-1)+(6k+1)+(6k+4) \\
&= 6k^2-k+12k+5 \\
&= 6k^2+11k+5 \\
&= (k+1)(6k+5) \\
&= (k+1)(6(k+1)-1).
\end{aligned}$$

Thus P_{k+1} holds and therefore, by PMI, $\sum_{i=1}^{2n} 3i-2 = n(6n-1)$ holds for any integer, $n \geq 1$.

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2. (a) [3 points] For any integer k, we introduce the notation $k! = k \cdot (k-1)(k-2)\cdots 2$, which we call "factorial k". Using this, write

$$1 + \frac{1}{2} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3 \cdot 2} + \cdots + \frac{1}{n \cdot (n-1) \cdots 2}$$

in sigma notation (do not evaluate).

Solution:

$$\sum_{i=1}^{n} \frac{1}{i!}$$

3. Write the following sums using sigma notation with indexes starting at 1 (do not evaluate):

(a) [3 points] $\frac{1}{6} + \frac{1}{3} + \frac{2}{3} + \frac{4}{3} + \frac{8}{3} + \cdots + \frac{64}{3}$

Solution: $\sum_{i=1}^{8} \frac{2^{i-1}}{6} = \sum_{i=1}^{8} \frac{2(2^{i-2})}{6} = \sum_{i=1}^{8} \frac{2^{i-2}}{3}$

(b) [4 points] $-x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$

Solution: $\sum_{i=1}^{5} (-1)^i \frac{x^i}{i!} = \sum_{i=1}^{5} \frac{(-1)^i x^i}{i!} = \sum_{i=1}^{5} \frac{(-x)^i}{i!}$

4. Let $f(x) = x^5 - 2x^3 + 2x^2 - 3x + 2$. Then $f(-x) = -x^5 + 2x^3 + 2x^2 + 3x + 2$.

(a) [4 points] Given that i is a root of $f(x)$, write $f(x)$ in the form $f(x) = P_2(x)Q_3(x)$, where $P_2(x)$ is an irreducible quadratic form and $Q_3(x)$ is a cubic polynomial.

Solution: Since i is a zero of $f(x)$, and all the coefficients of f are real, we know that $-i$ is also a zero. Thus $(x - i)(x + i) = x^2 + 1$ is a factor. Dividing out we get

$$f(x) = (x^2 + 1)(x^3 - 3x + 2).$$

(b) [4 points] What do Descartes' Rules of Signs say about $f(x)$? Be specific, and use complete sentences.

Solution: Since $f(x)$ has 4 sign changes, we know that $f(x)$ has either 4, 2, or 0 positive zeros. Since $f(-x)$ has 1 sign change, we know that $f(x)$ has 1 negative zero.

(c) [3 points] What does the Bounds Theorem say about $f(x)$? Be specific, and use complete sentences.

Solution: If x is a zero of $f(x)$, then

$$|x| < \frac{\max\{|-2|, |2|, |-3|, |2|\}}{|a_5|} + 1 = \frac{3}{1} + 1 = 4.$$

(d) [6 points] Taking the results of (a), (b) and (c) into account, use the rational root theorem to list all possible rational zeros of $f(x)$. Finish by listing all the zeros of $f(x)$ together with their multiplicity if it is not equal to 1.

Solution: If p/q is a rational root of $f(x) = 0$ (in lowest terms), then $p \mid 2$ and $q \mid 1$, and thus

$$\frac{p}{q} \in \{\pm 1, \pm 2\}.$$

By inspection, $Q_3(1) = 0$, and dividing out we get $Q_3(x) = x^3 - 3x + 2 = (x - 1)(x^2 + x - 2) = (x - 1)(x - 1)(x + 2)$. Therefore $f(x) = (x^2 + 1)(x - 1)^2(x + 2)$ and so the zeros of $f(x)$ are:

$$i, \quad -i, \quad 1 \text{ (multiplicity 2), and} \quad -2.$$

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5. [3 points] Let $g(x) = 5x^5 + 4x^4 + kx^3 + 2x^2 + x + 1$. For what value(s) of k will $g(x)$ have remainder 15 when divided by $x - 1$?

Solution: $15 = g(1) = 5 + 4 + k + 2 + 1 + 1 = k + 13$ and therefore $k = 2$.

6. (a) [3 points] Express $z_1 = \sqrt{2} + \sqrt{2}i$ in exponential form:

Solution: $r = \sqrt{2+2} = \sqrt{4} = 2$, $\cos\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}$. Thus

$$z_1 = 2e^{\frac{\pi}{4}i}.$$

(b) [3 points] Express $z_2 = 6\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$ in Cartesian form.

Solution:

$$z_2 = 6\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) = 6\left(-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right) = 6\left(\frac{-\sqrt{3}+i}{2}\right) = -3\sqrt{3} + 3i.$$

(c) [3 points] Express $z_3 = -7e^{\frac{\pi}{3}i}$ in polar form.

Solution:

$$z_3 = -7e^{\frac{\pi}{3}i} = -7(\cos\pi/3 + i\sin\pi/3) = 7(\cos 4\pi/3 + i\sin 4\pi/3).$$

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7. Consider the complex number $z = 27(1-i)$.

(a) [3 points] Write z in polar and exponential forms.

Solution:

$$z = 27 - 27i$$

$$r^2 = 27^2 + (-27)^2 = 2(27)^2 \implies r = 27\sqrt{2}$$

$$\theta = -\pi/4$$

$$z = 27\sqrt{2}(\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)) = 27\sqrt{2}e^{\left(\frac{-\pi}{4}\right)i} = 27\sqrt{2}e^{\left(\frac{-\pi}{4}+2k\pi\right)i} = 27\sqrt{2}e^{\frac{(8k-1)\pi}{4}i}$$

(b) [6 points] Find all cube roots of z. You can leave your answers in exponential form.

Solution: Let $x^3 = z = 27\sqrt{2}e^{\frac{(8k-1)\pi}{4}i}$. Then

$$x = \left(27\sqrt{2}e^{\frac{(8k-1)\pi}{4}i}\right)^{1/3} = 3(2^{1/6})e^{\frac{(8k-1)\pi}{12}i}$$

$$k = 0 : x_1 = 3(2^{1/6})e^{\frac{-\pi}{12}i}$$
$$k = 1 : x_2 = 3(2^{1/6})e^{\frac{7\pi}{12}i}$$
$$k = 2 : x_3 = 3(2^{1/6})e^{\frac{15\pi}{12}i} = 3(2^{1/6})e^{\frac{5\pi}{4}i}.$$

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8. Consider the following matrices:

$$A = \begin{bmatrix} -4 & -1 & -5 \end{bmatrix}, \qquad B = \begin{bmatrix} -9 & -1 & 0 \\ 0 & 6 & -5 \end{bmatrix}, \qquad C = \begin{bmatrix} 4 & -2 & -5 \\ 7 & 8 & 2 \\ 8 & -5 & -3 \end{bmatrix},$$

$$D = \begin{bmatrix} 6 & 0 & -3 \\ 4 & 2 & -5 \\ 3 & -1 & -6 \\ 5 & 1 & -2 \end{bmatrix}, \qquad E = \begin{bmatrix} 4 & -4 \\ -4 & -1 \\ 0 & -8 \\ -2 & 7 \end{bmatrix}, \qquad F = \begin{bmatrix} -4 & -5 \\ -4 & 8 \\ -10 & 1 \end{bmatrix}.$$

(a) [8 points] For each expression below, indicate with an ″X″ in the appropriate column if it is undefined or defined, and if it is defined, indicate the size of the resulting matrix. Do not compute the resulting matrix.

Expression	Undefined	Defined	Size
ACF		X	1×2
$F^T F + I_2$		X	2×2
$D^T E - B$	X		
$AA^T - A^T A$	X		
$DC^T - D$		X	4×3

(b) [2 points] Evaluate the $(2,3)$ entry of $B^T F^T - C$ if it is defined. If it is not defined, explain why.

Solution: The $(2,3)$ entry of $B^T F^T = (FB)^T$ is the $(3,2)$ entry of FB:

$$\begin{bmatrix} -10 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \end{bmatrix} = 10 + 6 = 16.$$

Therefore the $(2,3)$ entry of $B^T F^T - C$ is $16 - C_{2,3} = 16 - 2 = 14$.

SCRAP / EXTRA PAPER

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