**MATH 1210 Assignment 1 Fall 2016** **Due date: October 3**

**1. Use mathematical induction on positive integer $n$ to prove each of the following:**

**(a)** $1(4)+2(7)+3(10)+\cdots+(n-1)(3n-2) = n^2(n-1)$, **for** $n \geq 2$ **;**

**(b)** $n!+2^n < (2n)!$ **for** $n \geq 2$**;**

**(c)** $(1^3+4^3)+(2^3+8^3)+(3^3+12^3)+\cdots+[n^3+(4n)^3] = \dfrac{65}{4}n^2(n+1)^2$ **for** $n \geq 1$**.**

**(d)** $(n+1)^3+2n+5$ **is divisible by** $3$**, for** $n \geq 0$**.**

**Solution:**

**(a) Let $P(n)$ be the statement $1(4)+2(7)+3(10)+\cdots+(n-1)(3n-2) = n^2(n-1)$.**
**If $n=2$, then $P(2)$ is true because $1(4)=4$ and $2^2(2-1)=4$.**
**We assume that for $n=k$, $P(k)$ is valid that is**

$$1(4)+2(7)+3(10)+\cdots+(k-1)(3k-2) = k^2(k-1) \qquad (*)$$

**We need to prove that for $n=k+1$, $P(k+1)$ is valid that is**

$$1(4)+2(7)+3(10)+\cdots+(k)(3k+1) = (k+1)^2(k)\,.$$

**But**

$$\begin{aligned}1(4)+2(7)+3(10)+\cdots+(k)(3k+1) &= [1(4)+2(7)+3(10)+\cdots+(k-1)(3k-2)]+(k)(3k+1)\\ &= k^2(k-1)+(k)(3k+1) \quad \textbf{by } (*)\\ &= k\left(k(k-1)+3k+1\right)\\ &= k(k^2+2k+1)\\ &= k(k+1)^2\,.\end{aligned}$$

**Hence $1(4)+2(7)+3(10)+\cdots+(k)(3k+1) = (k+1)^2(k)$.**
**Therefore by the principle of mathematical induction $P(n)$ is valid for all $n \geq 2$.**

**(b) Let $P(n)$ be the statement $n!+2^n < (2n)!$.**
**If $n=2$, then $P(2)$ is true because $2!+2^2 = 6 < 4! = 24$. We assume that for $n=k$, $P(k)$ is valid that is $k!+2^k < (2k)!$ We need to prove that for $n=k+1$, $P(k+1)$ is valid that is $(k+1)!+2^{k+1} < (2k+2)!$. But**

$$\begin{aligned}(k+1)!+2^{k+1} &= (k+1)\,k! + 2^{k+1} - (k+1)2^k + (k+1)2^k\\ &= (k+1)(k!+2^k) - 2^k((k+1)-2)\\ &< (k+1)(2k)! - 2^k(k-1) \quad \textbf{by the induction hypothesis}\\ &< (k+1)(2k)!\\ &< 2(k+1)(2k)!\\ &< (2k+2)(2k+1)(2k)!\\ &= (2k+2)!\,.\end{aligned}$$

**Hence $(k+1)!+2^{k+1} < (2k+2)!$.**
**Therefore by the principle of mathematical induction $P(n)$ is valid for all $n \geq 2$.**

**(c) Let $P(n)$ be the statement " $(1^3+4^3)+(2^3+8^3)+(3^3+12^3)+\cdots+[n^3+(4n)^3] = \frac{65}{4}n^2(n+1)^2$ ".**

**If $n=1$, then $P(1)$ is true because $1^3+4^3=65$ and $\frac{65}{4}1^2(1+1)^2=65$. We assume that for $n=k$, $P(k)$ is valid that is**

$$(1^3+4^3)+(2^3+8^3)+(3^3+12^3)+\cdots+[k^3+(4k)^3] = \frac{65}{4}k^2(k+1)^2.$$

**We need to prove that for $n=k+1$, $P(k+1)$ is valid that is**

$$(1^3+4^3)+(2^3+8^3)+(3^3+12^3)+\cdots+[(k+1)^3+(4k+4)^3] = \frac{65}{4}(k+1)^2(k+2)^2.$$

**But**

$$\begin{aligned}
&(1^3+4^3)+(2^3+8^3)+(3^3+12^3)+\cdots+[(k+1)^3+(4k+4)^3]\\
&= \big((1^3+4^3)+(2^3+8^3)+(3^3+12^3)+\cdots+[k^3+(4k)^3]\big) + [(k+1)^3+(4k+4)^3]\\
&= \frac{65}{4}k^2(k+1)^2 + [(k+1)^3+4^3(k+1)^3] \quad \textbf{by the induction hypothesis}\\
&= (k+1)^2\,[\frac{65}{4}k^2+(k+1)+64(k+1)]\\
&= (k+1)^2\,[\frac{65}{4}k^2+65k+65]\\
&= \frac{65}{4}(k+1)^2[k^2+4k+4]\\
&= \frac{65}{4}(k+1)^2(k+2)^2\,.
\end{aligned}$$

**Therefore by the principle of mathematical induction $P(n)$ is true for all $n \geq 1$.**

**(d) Let $P(n)$ be the statement $(n+1)^3+2n+5$ is divisible by $3$.**
**If $n=0$, then $P(0)$ is true because $(0+1)^3+2(0)+5=6$ which is divisible by $3$. We assume that for $n=k$, $P(k)$ is valid that is $(k+1)^3+2k+5$ is divisible by $3$. We need to prove that for $n=k+1$, $P(k+1)$ is valid that is $(k+2)^3+2k+7$ is divisible by $3$. But**

$$\begin{aligned}
(k+2)^3+2k+7 &= k^3+6k^2+12k+8+2k+7\\
&= [(k^3+3k^2+3k+1)\;+\;2k+5]\;+\;(3k^2+9k+9)\\
&= [(k+1)^3\;+\;2k+5]\;+\;3(k^2+3k+3)\,.
\end{aligned}$$

**Since $(k+1)^3\;+\;2k+5$ is divisible by $3$ (by the induction hypothesis) and also $3(k^2+3k+3)$ is divisible by $3$ so $(k+2)^3+2k+7$ must be divisible by $3$.**
**Therefore by the principle of mathematical induction $P(n)$ is valid for all $n \geq 0$.**

**2. Consider the sum $(3)^2+(7)^2+(11)^2+\cdots+(12n-1)^2$:**

**(a) Write the sum in sigma notation.**

**Solution: Since $3=4(1)-1$, $7=4(2)-1$ and $(12n-1)=4(3n)-1$ so**

$$(3)^2+(7)^2+(11)^2+\cdots+(12n-1)^2 = \sum_{j=1}^{3n}(4j-1)^2\,.$$

**(b) Use identities** $\sum_{k=1}^{m} k = \frac{1}{2}\,[m(m+1)]$ **and** $\sum_{k=1}^{m} k^2 = \frac{1}{6}\,[m(m+1)(2m+1)]$ **to prove that**

$$(3)^2 + (7)^2 + (11)^2 + \cdots + (12n-1)^2 = n\,(144n^2 + 36n - 1)\,.$$

**Solution:**

$$\begin{aligned}
(3)^2 + (7)^2 + (11)^2 + \cdots + (12n-1)^2 &= \sum_{j=1}^{3n}(4j-1)^2 \\
&= \sum_{j=1}^{3n}(16j^2 - 8j + 1) \\
&= 16\sum_{j=1}^{3n} j^2 - 8\sum_{j=1}^{3n} j + \sum_{j=1}^{3n} 1 \\
&= 16\,[\frac{1}{6}(3n)(3n+1)(6n+1)] - 8[\frac{1}{2}(3n)(3n+1)] + 3n \\
&= n\,[8(3n+1)(6n+1) - 12(3n+1) + 3] \\
&= n\,(144n^2 + 36n - 1)\,.
\end{aligned}$$

**3. First write the sum** $n + (n+2) + (n+4) + (n+6) + \cdots + (3n)$ **in sigma notation and then use the identity** $\sum_{k=1}^{m} k = \frac{1}{2}\,[m(m+1)]$ **to find the value of the sigma in terms of** $n$**.**

**Solution:**

$$n + (n+2) + (n+4) + (n+6) + \cdots + (3n) \;=\; \sum_{k=0}^{n}(n+2k)$$

**Now** $\sum_{k=0}^{n} n = n\sum_{k=0}^{n} 1 = n(n+1)$ **and also**

$$\sum_{k=0}^{n}(2k) = 2\sum_{k=0}^{n} k = 2\sum_{k=1}^{n} k = 2(\frac{1}{2})n(n+1) = n(n+1)\,.$$

**Therefore**

$$\sum_{k=0}^{n}(n+2k) = \sum_{k=0}^{n} n \;+\; \sum_{k=0}^{n}(2k) = n(n+1) \;+\; n(n+1) = 2n(n+1)\,.$$

**4. Prove that** $\sum_{\ell=1}^{n} \ell(\ell-3) = \frac{1}{3}\,[n(n+1)(n-4)]$ **by each of the following two methods:**

**(a)** **By mathematical induction on positive integer** $n \geq 1$ **.**

**(b)** **By using the identities mentioned in part (b) of question** $2$ **.**

**Solution:**

**(a)** **Let** $P(n)$ **be the statement** $\displaystyle\sum_{\ell=1}^{n} \ell(\ell-3) = \frac{1}{3}\left[n(n+1)(n-4)\right]$.

**If** $n = 1$, **then** $P(1)$ **is true because** $\displaystyle\sum_{\ell=1}^{1} \ell(\ell-3) = 1(1-3) = -2$ **and also** $\frac{1}{3}[1(1+1)(1-4)] = -2$.

**We assume that for** $n = k$, $P(k)$ **is valid that is** $\displaystyle\sum_{\ell=1}^{k} \ell(\ell-3) = \frac{1}{3}\left[k(k+1)(k-4)\right]$.
**We need to prove that for** $n = k+1$, $P(k+1)$ **is valid that is**

$$\sum_{\ell=1}^{k+1} \ell(\ell-3) \;=\; \frac{1}{3}\big[(k+1)(k+2)(k-3)\big]\,.$$

**But**

$$\begin{aligned}
\sum_{\ell=1}^{k+1} \ell(\ell-3) \;&=\; \sum_{\ell=1}^{k} \ell(\ell-3) \;+\; (k+1)(k-2) \\
&= \frac{1}{3}[k(k+1)(k-4)] \;+\; (k+1)(k-2) \quad \textbf{by the induction hypothesis} \\
&= \frac{1}{3}\big[k(k+1)(k-4) \;+\; 3(k+1)(k-2)\big] \\
&= \frac{1}{3}\Big[(k+1)\big(k(k-4) \;+\; 3(k-2)\big)\Big] \\
&= \frac{1}{3}\big[(k+1)(k^2-k-6)\big] \\
&= \frac{1}{3}\big[(k+1)(k+2)(k-3)\big]\,.
\end{aligned}$$

**Therefore by the principle of mathematical induction** $P(n)$ **is valid for all** $n \geq 1$.

**(b)** **Using the given formulas we have**

$$\begin{aligned}
\sum_{\ell=1}^{n} \ell(\ell-3) \;=\sum_{\ell=1}^{n} \ell^2 \;-\; 3\sum_{\ell=1}^{n} \ell = \;& \frac{1}{6}[n(n+1)(2n+1)] \;-\; 3\left[\frac{1}{2}\big(n(n+1)\big)\right] \\
=\;& \frac{1}{6}\big[n(n+1)(2n+1) \;-\; 9n(n+1)\big] \\
=\;& \frac{1}{6}\Big[n(n+1)\big((2n+1) \;-\; 9\big)\Big] \\
=\;& \frac{1}{6}\Big[n(n+1)\big(2(n-4)\big)\Big] \\
=\;& \frac{1}{3}[n(n+1)(n-4)]\,.
\end{aligned}$$

**5. Rewrite the sum** $\displaystyle\sum_{j=-4}^{50} \big[(4j+20)^3 + j(j+10) + 25\big]$ **such that it starts with** $j = 1$. **Simplify your answer but keep it in sigma notation.**

**Solution: First we notice that** $\sum_{j=-4}^{50}\big[(4j+20)^3+j(j+10)+25\big] = \sum_{j=-4}^{50}\big[(4j+20)^3+(j+5)^2\big]$ **.**
**Now if we replace** $j$ **by** $j-5$**, then**

$$\begin{aligned}\sum_{j=-4}^{50}\big[(4j+20)^3+j(j+10)+25\big] &= \sum_{j=-4}^{50}\big[(4j+20)^3+(j+5)^2\big]\\ &= \sum_{j=1}^{55}\Big[\big(4(j-5)+20\big)^3+\big((j-5)+5\big)^2\Big]\\ &= \sum_{j=1}^{55}\big[(4j)^3\,+\,j^2\big]\\ &= 64\sum_{j=1}^{55} j^3\,+\,\sum_{j=1}^{55} j^2\,.\end{aligned}$$

**6. Find values of** $x$ **and** $y$ **if**

$$\sum_{j=4}^{14}[(4j+1)^{10}\,-\,(2j+6)^2]\,=\,\sum_{j=7}^{17}[(4j+x)^{10}\,+\,yj^2]\,.$$

**Solution: If we replace** $j$ **by** $j-3$ **, then**

$$\sum_{j=4}^{14}[(4j+1)^{10}\,-\,(2j+6)^2]\,=\,\sum_{j=7}^{17}\Big[\big(4(j-3)+1\big)^{10}\,+\,\big(2(j-3)+6\big)^2\Big]\,=\,\sum_{j=7}^{17}[(4j-11)^{10}\,-\,4j^2]\,.$$

**Now comparing with** $\sum_{j=7}^{17}[(4j+x)^{10}\,+\,yj^2]$ **gives** $x=-11$ **and** $y=-4$

**7. Find all real and complex solutions of the equation**

$$x^5+x^4+3x^3-x^2-x-3=0\,.$$

**Solution:**

$$\begin{aligned}x^5+x^4+3x^3-x^2-x-3 &= (x^5-x^2)\,+(x^4-x)\,+(3x^2-3)\\ &= x^2(x^3-1)\,+x(x^3-1)\,+3(x^3-1)\\ &= (x^3-1)(x^2+x+3)\\ &= (x-1)(x^2+x+1)(x^2+x+3)\,.\end{aligned}$$

**Now** $(x-1)(x^2+x+1)(x^2+x+3)\,=\,0$ **implies either** $x-1=0$ **so** $x=1$**, or** $x^2+x+1=0$ **so** $x=\frac{1}{2}\pm\frac{\sqrt{3}}{2}\,i$**, and or** $x^2+x+3=0$ **so** $x=\frac{1}{2}\pm\frac{\sqrt{11}}{2}\,i$ **.**

**8. Find the Cartesian form of each of the following expression. Simplify as much as possible.**

**(a)** $\left(\frac{\sqrt{3}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i\right)^{12} - \left(\frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i\right)^{14}$**.**

**Solution: We can use either exponential form or polar form. Here we use polar form. Let** $z_1 = \frac{\sqrt{3}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i$ **and** $z_2 = \frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i$ **.Then**

$$r_1 = \sqrt{(\frac{\sqrt{3}}{\sqrt{2}})^2 + (\frac{-1}{\sqrt{2}})^2} = \sqrt{\frac{3}{2} + \frac{1}{2}} = \sqrt{2}\,;$$

**and** $r_2 = \sqrt{(\frac{\sqrt{3}}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2} = \sqrt{2}$**. Also**

$$\tan\,\theta_1 = \frac{-\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{\sqrt{2}}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \Rightarrow \theta_1 = -\frac{\pi}{6}\,.$$

**Similarly** $\tan\,\theta_2 = \frac{\sqrt{3}}{3}$ **that is** $\theta_2 = \frac{\pi}{6}$**. So** $z_1 = \sqrt{2}\left(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right)$ **and** $z_2 = \sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$**. Using DeMoivre's identity we get**

$$\begin{aligned}
z_1^{12} - z_2^{14} &= [\sqrt{2}\left(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right)]^{12} - [\sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)]^{14} \\
&= (\sqrt{2})^{12}[\cos(-\frac{12\pi}{6}) + i\sin(-\frac{12\pi}{6})] - (\sqrt{2})^{14}[\cos\frac{14\pi}{6} + i\sin\frac{14\pi}{6}] \\
&= (2^6)[\cos(-2\pi) + i\sin(-2\pi)] - (2^7)[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}] \\
&= (2^6)[1 + i(0)] - (2^7)[\frac{1}{2} + \frac{\sqrt{3}}{2}\,i] \\
&= 2^6 - 2^6 - 2^6\sqrt{3}\,i \\
&= -64\sqrt{3}\,i\,.
\end{aligned}$$

**(b)** $\frac{(\overline{-1+i})^{13}\,(3+3\sqrt{3}\,i)^6}{6^6\,i^3(1-i)}$**.**

**Solution: We first note that** $i^3(1-i) = -i(1-i) = -i + i^2 = -1 - i$ **.**
**Since** $|-1-i| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$ **and** $\tan\theta = \frac{-1}{-1} = 1$ **so** $\theta = -\frac{3\pi}{4}$ **(we could use** $\theta = \frac{5\pi}{4}$ **). Therefore** $-1 - i = \sqrt{2}\,e^{-\frac{3\pi}{4}i}$**. Also**

$$|3 + 3\sqrt{3}\,i| = |3||1 + \sqrt{3}\,i| = 3\sqrt{1^2 + (\sqrt{3})^2} = 3(2) = 6\,;$$

**and** $\tan\theta = \frac{3\sqrt{3}}{3} = \sqrt{3}$ **so** $\theta = \frac{\pi}{3}$ **Therefore** $3 + 3\sqrt{3}\,i = 6e^{\frac{\pi}{3}i}$**.**

**Also** $\overline{-1+i} = -1-i = \sqrt{2}\, e^{-\frac{3\pi}{4} i}$**. Now**

$$\begin{aligned}
\frac{(\overline{-1+i})^{13}\,(3+3\sqrt{3}\,i)^6)^6}{6^6\, i^3(1-i)} &= \frac{(-1-i)^{13}\,(3+3\sqrt{3}\,i)^6}{6^6\,(-1-i)} \\
&= \frac{(-1-i)^{12}\,(3+3\sqrt{3}\,i)^6}{6^6} \\
&= \frac{(\sqrt{2}\, e^{-\frac{3\pi}{4} i})^{12}\,(6e^{\frac{\pi}{3} i})^6}{6^6} \\
&= \frac{(2^6)[e^{-\frac{36\pi}{4} i}]\;(6^6)[e^{\frac{6\pi}{3} i}]}{6^6} \\
&= (2^6)[e^{-9\pi i}]\,[e^{2\pi i}] \\
&= 2^6[e^{-7\pi i}] \\
&= 64[e^{-\pi i}] \\
&= 64[\cos(-\pi) + i\sin(-\pi)] \\
&= 64[(-1) + i(0)] \\
&= -64\,.
\end{aligned}$$