**MATH 1210 Assignment 3 (Section A02) Fall 2016** **Due date: December 12**

**Attempt all questions and show all your work. Attach to Honesty Declaration Form.**

1. **For each of the following linear system of equations, first find reduced row echelon form of the augmented matrix and then find all solutions of the system.**

$$(i)\quad \begin{aligned} x \qquad +z-w &= 1\\ -x+y \quad -z+w &= 0\\ 2y \quad +z+w &= 1\\ 3y \quad +z+w &= 2 \end{aligned} \qquad\qquad (ii)\quad \begin{aligned} 3x+5y&=1\\ 2x+y&=3\\ 4x+2y&=6\\ 5x+6y&=4 \end{aligned}$$

$$(iii)\quad \begin{aligned} x_1 \qquad -x_3-x_4 &= 5\\ -2x_1+x_2 \quad +3x_3+4x_4 &= -2\\ 3x_1+x_2 \quad -2x_3-x_4 &= 9 \end{aligned} \qquad\qquad (iv)\quad \begin{aligned} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}&=4\\ \frac{2}{x}-\frac{3}{y}-\frac{1}{z}&=1\\ -\frac{1}{x}+\frac{2}{y}+\frac{1}{z}&=0 \end{aligned}$$

**Solution:**

**(a) (i)**

$$\left[\begin{array}{cccc|c} 1&0&1&-1&1\\ -1&1&-1&1&0\\ 0&2&1&1&1\\ 0&3&1&1&2 \end{array}\right] R_2\to R_1+R_2 \Rightarrow \left[\begin{array}{cccc|c} 1&0&1&-1&1\\ 0&1&0&0&1\\ 0&2&1&1&1\\ 0&3&1&1&2 \end{array}\right] \begin{array}{l} \Rightarrow \\ R_3\to -2R_2+R_3\\ R_4\to -3R_2+R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1&0&1&-1&1\\ 0&1&0&0&1\\ 0&0&1&1&-1\\ 0&0&1&1&-1 \end{array}\right] \begin{array}{l} R_1\to -R_3+R_1\\ \\ R_4\to -R_3+R_4 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1&0&0&-2&2\\ 0&1&0&0&1\\ 0&0&1&1&-1\\ 0&0&0&0&0 \end{array}\right] RREF \Rightarrow$$

$$\begin{aligned} x-2w&=2\\ y&=1\\ z+w&=-1\\ 0&=0 \end{aligned} \quad\Rightarrow\quad \begin{aligned} x&=2+2t\\ y&=1\\ z&=-1-t\\ w&=t \end{aligned} \qquad t\in\mathbf{R}$$

**(b) (ii)**

$$\left[\begin{array}{cc|c} 3&5&1\\ 2&1&3\\ 4&2&6\\ 5&6&4 \end{array}\right] R_1\to R_2+R_1 \Rightarrow \left[\begin{array}{cc|c} 1&4&-2\\ 2&1&3\\ 4&2&6\\ 5&6&4 \end{array}\right] \begin{array}{l} R_2\to -2R_1+R_2\\ R_3\to -4R_1+R_3\\ R_4\to -5R_1+R_4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cc|c} 1&4&-2\\ 0&-7&7\\ 0&-14&14\\ 0&-14&14 \end{array}\right] R_2\to \tfrac{1}{-7}R_2 \Rightarrow \left[\begin{array}{cc|c} 1&4&-2\\ 0&1&-1\\ 0&-14&14\\ 0&-14&14 \end{array}\right] \begin{array}{l} R_1\to -4R_2+R_1\\ \\ R_3\to 14R_2+R_3\\ R_4\to 14R_2+R_4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cc|c} 1&0&2\\ 0&1&-1\\ 0&0&0\\ 0&0&0 \end{array}\right] RREF \Rightarrow \begin{aligned} x&=2\\ y&=-1 \end{aligned} \Rightarrow (2,-1).$$

**(c) (iii)**

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 5 \\ -2 & 1 & 3 & 4 & -2 \\ 3 & 1 & -2 & -1 & 9 \end{array}\right] \begin{array}{l} R_2 \to 2R_1 + R_2 \\ R_3 \to -3R_1 + R_3 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 5 \\ 0 & 1 & 1 & 2 & 8 \\ 0 & 1 & 1 & 2 & 6 \end{array}\right] R_3 \to -R_2 + R_3$$

$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 5 \\ 0 & 1 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 & -14 \end{array}\right]$. **Since from the last row we get** $0 = -14$ **which is not possible, so this system has no solution.**

**(d) (iv) Let** $\frac{1}{x} = x_1$ **,** $\frac{1}{y} = x_2$ **and** $\frac{1}{z} = x_3$ **. Then it becomes** $\begin{array}{rr} x_1 + x_2 & +x_3 = 4 \\ 2x_1 - 3x_2 & -x_3 = 1 \\ -x_1 + 2x_2 & +x_3 = 0 \end{array}$**. So**

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & -3 & -1 & 1 \\ -1 & 2 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 \to -2R_1 + R_2 \\ R_3 \to R_1 + R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -5 & -3 & -7 \\ 0 & 3 & 2 & 4 \end{array}\right] R_3 \to -R_2 + R_3 \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -5 & -3 & -7 \\ 0 & 3 & 2 & 4 \end{array}\right] R_2 \to \tfrac{1}{-5}R_2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & 3 & 2 & 4 \end{array}\right] \begin{array}{l} R_1 \to -R_2 + R_1 \\ \\ R_3 \to -3R_2 + R_3 \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{5} & \frac{13}{5} \\ 0 & 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & \frac{1}{5} & -\frac{1}{5} \end{array}\right] R_3 \to 5R_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{5} & \frac{13}{5} \\ 0 & 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 \to -\frac{2}{5}R_3 + R_1 \\ R_2 \to -\frac{3}{5}R_3 + R_2 \end{array} \Rightarrow$$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$. **So** $x_1 = 3$ **,** $x_2 = 2$ **and** $x_3 = -1$ **; which means** $x = \frac{1}{3}$ **,** $y = \frac{1}{2}$ **and** $z = -1$ **. Therefore the solution of the system is** $(\frac{1}{3}, \frac{1}{2}, -1)$ **.**

**2. Find all *basic* solutions of the homogeneous system**

$$\begin{aligned} 9x_1 + 9x_2 + 4x_3 - 4x_4 + 9x_5 - 10x_6 &= 0 \\ 2x_1 + 2x_2 + x_3 - x_4 + 2x_5 - 3x_6 &= 0 \\ 11x_1 + 11x_2 + 3x_3 + 4x_4 + 4x_5 + 22x_6 &= 0 \end{aligned}$$

**Solution:**

$$\left[\begin{array}{cccccc|c} 9 & 9 & 4 & -4 & 9 & -10 & 0 \\ 2 & 2 & 1 & -1 & 2 & -3 & 0 \\ 11 & 11 & 3 & 4 & 4 & 22 & 0 \end{array}\right] R_1 \to -4R_2 + R_1 \Rightarrow \left[\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 1 & 2 & 0 \\ 2 & 2 & 1 & -1 & 2 & -3 & 0 \\ 11 & 11 & 3 & 4 & 4 & 22 & 0 \end{array}\right] \begin{array}{l} R_2 \to -2R_1 + R_2 \\ R_3 \to -11R_1 + R_3 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & -7 & 0 \\ 0 & 0 & 3 & 4 & -7 & 0 & 0 \end{array}\right] R_3 \to -3R_2 + R_3 \Rightarrow \left[\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & -7 & 0 \\ 0 & 0 & 0 & 7 & -7 & 21 & 0 \end{array}\right] R_3 \to \tfrac{1}{7}R_3 \Rightarrow$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & -7 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & 0 \end{array}\right] R_2 \to R_3 + R_2 \Rightarrow \left[\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & -4 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & 0 \end{array}\right] RREF \Rightarrow$$

$$\begin{aligned} x_1 + x_2 + x_5 + 2x_6 &= 0 \\ x_3 - x_5 - 4x_6 &= 0 \\ x_4 - x_5 + 3x_6 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= - x_2 - x_5 - 2x_6 \\ x_3 &= x_5 + 4x_6 \\ x_4 &= x_5 - 3x_6 \end{aligned}$$

**Let** $x_2 = t$ **,** $x_5 = r$ **and** $x_6 = s$ **. Then** $\begin{aligned} x_1 &= - t - r - 2s \\ x_2 &= t \\ x_3 &= r + 4s \\ x_4 &= r - 3s \\ x_5 &= r \\ x_6 &= s \end{aligned}$ **where** $t, r, s \in \mathbf{R}$ **. But**

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -t - r - 2s \\ t \\ r + 4s \\ r - 3s \\ r \\ s \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -r \\ 0 \\ r \\ r \\ r \\ 0 \end{bmatrix} + \begin{bmatrix} -2s \\ 0 \\ 4s \\ -3s \\ 0 \\ s \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 4 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

**Therefore a basic solution is** $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$**.**

**3. For any real number** $n$ **let** $A = \begin{pmatrix} n+1 & n+2 & n+3 \\ n+4 & n+5 & n+6 \\ n+7 & n+8 & n+9 \end{pmatrix}$ **. Use properties of determinant to find** $|A|$**. (Explain)**

**Solution:**

$$|A| = \begin{vmatrix} n+1 & n+2 & n+3 \\ n+4 & n+5 & n+6 \\ n+7 & n+8 & n+9 \end{vmatrix} \begin{matrix} \\ R_2 \to -R_1 + R_2 \\ R_3 \to -R_1 + R_3 \end{matrix} = \begin{vmatrix} n+1 & n+2 & n+3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} R_3 \to -2R_2 + R_3$$

$$= \begin{vmatrix} n+1 & n+2 & n+3 \\ 3 & 3 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

**4. Let** $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ **. Find** $det(-10A)$ **.**

**Solution: First** $det(-10A) = (-10)^4 det(A) = 10000\, det(A)$ **. But**

$$det(A) = \begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix} = +1 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \left( 1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \right)$$

$$-\left(1\begin{vmatrix}1 & 1\\1 & 1\end{vmatrix} + 1\begin{vmatrix}1 & 1\\0 & 1\end{vmatrix}\right) + \left(1\begin{vmatrix}0 & 1\\1 & 1\end{vmatrix} - 1\begin{vmatrix}1 & 1\\1 & 1\end{vmatrix}\right) = (0+(-1)) - (0+1) + (-1-0) = -3.$$

**Therefore** $det(-10A) = 10000(-3) = -30000$ **.**

**5. Use Cramer's rule to solve each of the following linear system of equations.**

**(a)**
$$\begin{aligned} 3x_1 + 4x_2 \quad +4x_3 &= 11 \\ 4x_1 - 4x_2 \quad +6x_3 &= 11 \\ 6x_1 - 6x_2 \qquad\qquad &= 3 \end{aligned}$$

**(b)**
$$\begin{aligned} 3y + z - u &= 4 \\ y + z &= 0 \\ 2y - z + 4u &= 6 \\ x + 5z &= 1 \end{aligned}$$
**(For** $y$ ***only*****, so *do not* find** $x$, $z$ **and** $u$**.)**

**Solution:**

**(a)** $|A| = \begin{vmatrix}3 & 4 & 4\\4 & -4 & 6\\6 & -6 & 0\end{vmatrix} = +6\begin{vmatrix}4 & 4\\-4 & 6\end{vmatrix} - (-6)\begin{vmatrix}3 & 4\\4 & 6\end{vmatrix} = 6(40) + 6(2) = 252.$

$|A_1| = \begin{vmatrix}11 & 4 & 4\\11 & -4 & 6\\3 & -6 & 0\end{vmatrix} = +3\begin{vmatrix}4 & 4\\-4 & 6\end{vmatrix} - (-6)\begin{vmatrix}11 & 4\\11 & 6\end{vmatrix} = 3(40) + 6(22) = 252.$ **So**

$x_1 = \dfrac{|A_1|}{|A|} = \dfrac{252}{252} = 1$ **. Also**

$|A_2| = \begin{vmatrix}3 & 11 & 4\\4 & 11 & 6\\6 & 3 & 0\end{vmatrix} = +6\begin{vmatrix}11 & 4\\11 & 6\end{vmatrix} - 3\begin{vmatrix}3 & 4\\4 & 6\end{vmatrix} = 6(22) - 3(2) = 126.$ **So**

$x_2 = \dfrac{|A_2|}{|A|} = \dfrac{126}{252} = \dfrac{1}{2}$ **. Also**

$|A_3| = \begin{vmatrix}3 & 4 & 11\\4 & -4 & 11\\6 & -6 & 3\end{vmatrix} = 3\begin{vmatrix}-4 & 11\\-6 & 3\end{vmatrix} - 4\begin{vmatrix}4 & 11\\6 & 3\end{vmatrix} + 11\begin{vmatrix}4 & -4\\6 & -6\end{vmatrix} = 3(54) - 4(-54) + 11(0) = 378.$

**So**

$x_3 = \dfrac{|A_3|}{|A|} = \dfrac{378}{252} = \dfrac{3}{2}$ **. Therefore the solution of the system is** $(1, \frac{1}{2}, \frac{3}{2})$ **.**

**(b)** $|A| = \begin{vmatrix}0 & 3 & 1 & -1\\0 & 1 & 1 & 0\\0 & 2 & -1 & 4\\1 & 0 & 5 & 0\end{vmatrix} = -1\begin{vmatrix}3 & 1 & -1\\1 & 1 & 0\\2 & -1 & 4\end{vmatrix} = (-1)\left(-1\begin{vmatrix}1 & 1\\2 & -1\end{vmatrix} + 4\begin{vmatrix}3 & 1\\1 & 1\end{vmatrix}\right)$

$= (-1)(3+8) = -11$ **. Also**

$|A_2| = \begin{vmatrix}0 & 4 & 1 & -1\\0 & 0 & 1 & 0\\0 & 6 & -1 & 4\\1 & 1 & 5 & 0\end{vmatrix} = -1\begin{vmatrix}4 & 1 & -1\\0 & 1 & 0\\6 & -1 & 4\end{vmatrix} = (-1)\left(4\begin{vmatrix}1 & 0\\-1 & 4\end{vmatrix} + 6\begin{vmatrix}1 & -1\\1 & 0\end{vmatrix}\right)$

$= (-1)(16+6) = -22$ **. Therefore** $y = \dfrac{|A_2|}{|A|} = \dfrac{-22}{-11} = 2$ **.**

**6. Determine whether each set of vectors is linearly dependent or linearly independent.**

**(a)** $\{\mathbf{u}_1 =< 4, 5 >, \mathbf{u}_2 =< 3, 10 >, \mathbf{u}_3 =< -11, 20 >\}$

**(b)** $\{\mathbf{u}_1 =< 4, 3, 6 >, \mathbf{u}_2 =< 5, 7, 1 >, \mathbf{u}_3 =< -1, -4, 5 >\}$

**(c)** $\{\mathbf{u}_1 =< 1, 2, -2, 4 >, \mathbf{u}_2 =< 1, 3, -3, 4 >, \mathbf{u}_3 =< 1, 2, -1, 4 >\}$

**Solution:**

**(a)** **There are three vectors in** $\mathbf{R}^2$ **so** $m = 3 > 2 = n$ **therefore** $\{\mathbf{u}_1 , \mathbf{u}_2 , \mathbf{u}_3\}$ **is a linearly dependent set.**

**(b)**

$$\begin{vmatrix} 4 & 5 & -1 \\ 3 & 7 & -4 \\ 6 & 1 & 5 \end{vmatrix} = 4\begin{vmatrix} 3 & -4 \\ 6 & 5 \end{vmatrix} - 5\begin{vmatrix} 7 & -4 \\ 1 & 5 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 7 \\ 6 & 1 \end{vmatrix} = 4(39) - 5(39) - (-39) = 156 - 195 + 39 = 0$$

**Therefore** $\{\mathbf{u}_1 , \mathbf{u}_2 , \mathbf{u}_3\}$ **is a linearly dependent set.**

**(c)** **If** $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$ **then** $c_1 < 1, 2, -2, 4 > + c_2 < 1, 3, -3, 4 > + c_3 < 1, 2, -1, 4 > = < 0, 0, 0, 0 >$. **So** $< c_1 + c_2 + c_3 , 2c_1 + 3c_2 + 2c_3 , -2c_1 - 3c_2 - c_3 , 4c_1 + 4c_2 + 4c_3 > = < 0, 0, 0, 0 >$. **Hence**

$$\begin{aligned} c_1 + c_2 \quad +c_3 &= 0 \\ 2c_1 + 3c_2 \quad +2c_3 &= 0 \\ -2c_1 - 3c_2 \quad -c_3 &= 0 \\ 4c_1 + 4c_2 \quad +4c_3 &= 0 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 2 & 0 \\ -2 & -3 & -1 & 0 \\ 4 & 4 & 4 & 0 \end{array}\right] \begin{array}{l} R_2 \to -2R_1 + R_2 \\ R_3 \to 2R_1 + R_3 \\ R_4 \to -4R_1 + R_4 \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 \to -R_2 + R_1 \\ \\ R_3 \to R_2 + R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_3 \to -R_3 \quad \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 \to -R_3 + R_1 \quad \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] RREF$$

**Therefore** $c_1 = 0$ , $c_2 = 0$ **and** $c_3 = 0$, **which means** $\{\mathbf{u}_1 , \mathbf{u}_2 , \mathbf{u}_3\}$ **is a linearly independent set.**

**7. Let** $\mathbf{u}_1 =< 1, 2, 1, 3 >$, $\mathbf{u}_2 =< 1, 3, 1, 3 >$, $\mathbf{u}_3 =< 4, 0, 0, 0 >$ **and** $\mathbf{u}_4 =< 6, -9, -2, -6 >$. **Show that they are linearly dependent and then write** $\mathbf{u}_4$ **as a linear combination of** $\mathbf{u}_1$ , $\mathbf{u}_2$ **and** $\mathbf{u}_3$ .

**Solution: If** $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 = \mathbf{0}$ **then** $c_1 < 1, 2, 1, 3 > + c_2 < 1, 3, 1, 3 > + c_3 < 4, 0, 0, 0 > + c_4 < 6, -9, -2, -6 > = < 0, 0, 0, 0 >$. **So** $< c_1 + c_2 + 4c_3 + 6c_4 , 2c_1 + 3c_2 - 9c_4 , c_1 + c_2 - 2c_4 , 3c_1 + 3c_2 - 6c_4 > = < 0, 0, 0, 0 >$. **Hence**

$$\begin{aligned} c_1 + c_2 + 4c_3 + 6c_4 &= 0 \\ 2c_1 + 3c_2 - 9c_4 &= 0 \\ c_1 + c_2 - 2c_4 &= 0 \\ 3c_1 + 3c_2 - 6c_4 &= 0 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{cccc|c} 1 & 1 & 4 & 6 & 0 \\ 2 & 3 & 0 & -9 & 0 \\ 1 & 1 & 0 & -2 & 0 \\ 3 & 3 & 0 & -6 & 0 \end{array}\right] \begin{array}{l} R_2 \to -2R_1 + R_2 \\ R_3 \to -R_1 + R_3 \\ R_4 \to -3R_1 + R_4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 4 & 6 & 0 \\ 0 & 1 & -8 & -21 & 0 \\ 0 & 0 & -4 & -8 & 0 \\ 0 & 0 & -12 & -24 & 0 \end{array}\right] \begin{array}{c} R_1 \to -R_2 + R_1 \\ \\ R_3 \to \frac{1}{-4} R_3 \\ \\ \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 12 & 27 & 0 \\ 0 & 1 & -8 & -21 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -12 & -24 & 0 \end{array}\right] \begin{array}{c} R_1 \to -12R_3 + R_1 \\ R_2 \to 8R_3 + R_2 \\ \\ R_4 \to 12R_3 + R_4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & -5 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] RREF \quad \Rightarrow \quad \begin{aligned} c_1 + 3c_4 &= 0 \\ c_2 - 5c_4 &= 0 \\ c_3 + 2c_4 &= 0 \\ 0 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= -3t \\ c_2 &= 5t \\ c_3 &= -2t \\ c_4 &= t \end{aligned} \quad t \in \mathbf{R}$$

**Therefore $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and $\mathbf{u}_4$ are linearly dependent. Now let $t = 1$ then $-3\mathbf{u}_1 + 5\mathbf{u}_2 - 2\mathbf{u}_3 + \mathbf{u}_4 = \mathbf{0}$ which means $\mathbf{u}_4 = 3\mathbf{u}_1 - 5\mathbf{u}_2 + 2\mathbf{u}_3$ .**