

UNIVERSITY OF MANITOBA

DATE: October 26, 2017

MIDTERM
TITLE PAGE
TIME: 75 minutes

EXAMINATION: Techniques of Classical and Linear Algebra
COURSE: MATH 1210 EXAMINER: Moghaddam, Ramsey, Szestopalow

NAME: (Print in ink) Solutions

SIGNATURE: (in ink) _____
(I understand that cheating is a serious offense)

STUDENT NUMBER: _____

- A01 9:30–10:20 AM MWF (207 Buller) M. Szestopalow
 A02 1:30–2:20 PM MWF (100 St. Paul) G. I. Moghaddam
 A03 1:30–2:20 PM MWF (221 Wallace) C. Ramsey

INSTRUCTIONS TO STUDENTS:

This is a 75 minutes exam. Please show your work clearly.

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 6 pages of questions and also 1 blank page for rough work. Please check that you have all the pages. You may remove the blank page if you want, but be careful not to loosen the staple.

The value of each question is indicated in the left hand margin beside the statement of the question. The total value of all questions is 60 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	11	
2	10	
3	6	
4	15	
5	8	
6	10	
Total:	60	

- [8] 1. (a) Use mathematical induction on integer $n \geq 1$ to prove that

$$2!(2) + 3!(3) + 4!(4) + \dots + (n+1)!(n+1) = (n+2)! - 2.$$

Let $p(n)$ be the above statement.

For $n=1$, $p(1)$ is true because $2!(2) = 4$ and $(1+2)! - 2 = 6 - 2 = 4$.

Assume that for $n=k$, $p(k)$ is true i.e.

$$2!(2) + 3!(3) + \dots + (k+1)!(k+1) = (k+2)! - 2 \quad (*)$$

We need to prove that for $n=k+1$, $p(k+1)$ is true i.e

$$2!(2) + 3!(3) + \dots + (k+2)!(k+2) = (k+3)! - 2$$

$$\text{But } 2!(2) + 3!(3) + \dots + (k+2)!(k+2)$$

$$= [2!(2) + 3!(3) + \dots + (k+1)!(k+1)] + (k+2)!(k+2)$$

$$= (k+2)! - 2 + (k+2)!(k+2) \quad \text{by } (*)$$

$$= (k+2)!(1+k+2) - 2$$

$$= (k+3)!(k+2) - 2$$

$$= R.H.S.$$

Therefore by the principle of mathematical induction, $p(n)$

is true for all $n \geq 1$.

- [3] (b) Write $2!(2) + 3!(3) + 4!(4) + \dots + (n+1)!(n+1)$ in sigma notation such that the index starts from 0.

$$2!(2) + 3!(3) + \dots + (n+1)!(n+1) = \sum_{j=1}^n (j+1)!(j+1)$$

$$= \sum_{j=0}^{n-1} (j+2)!(j+2)$$

- [10] 2. Find the Cartesian form of $\frac{i^{62}(\sqrt{2} + \sqrt{6}i)^8}{2^8(-\sqrt{6} - \sqrt{2}i)}.$ Simplify as much as possible.

$$i^{62} = i^{60} \cdot i^2 = (i^4)^{15} (-1) = 1^{15} (-1) = -1$$

$$\text{Let } z = \sqrt{2} + \sqrt{6}i, \text{ then } |z| = \sqrt{(\sqrt{2})^2 + (\sqrt{6})^2} = \sqrt{2+6} = \sqrt{8}$$

$$\tan \theta = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3} \quad \text{so} \quad \theta = \frac{\pi}{3}; \text{ hence } z = \sqrt{8} e^{\frac{\pi}{3}i}$$

$$\text{Now } z^8 = (\sqrt{2} + \sqrt{6}i)^8 = \sqrt{8}^8 (e^{\frac{\pi}{3}i})^8 = 8^4 e^{\frac{8\pi}{3}i} = 8^4 e^{(2\pi + 2\frac{\pi}{3})i} = 8^4 e^{2\frac{5\pi}{3}i}$$

$$\text{So } z^8 = 8^4 (\cos^2 \frac{\pi}{3} + i \sin^2 \frac{\pi}{3}) = 2^{12} (-\frac{1}{2} + \frac{\sqrt{3}}{2}i) = 2^{11} (-1 + \sqrt{3}i)$$

$$\text{Therefore } \frac{i^{62}(\sqrt{2} + \sqrt{6}i)^8}{2^8(-\sqrt{6} - \sqrt{2}i)} = \frac{(-1)(2^{11})(-1 + \sqrt{3}i)}{2^8(-\sqrt{6} - \sqrt{2}i)}$$

$$= (-2^3) \frac{-1 + \sqrt{3}i}{-\sqrt{6} - \sqrt{2}i} \times \frac{-\sqrt{6} + \sqrt{2}i}{-\sqrt{6} + \sqrt{2}i}$$

$$= (-8) \frac{(-1 + \sqrt{3}i)(-\sqrt{6} + \sqrt{2}i)}{(-\sqrt{6})^2 + (-\sqrt{2})^2}$$

$$= (-8) \frac{\sqrt{6} - \sqrt{2}i - \sqrt{18}i + \sqrt{6}i}{6 + 2}$$

$$= (-8) \frac{\sqrt{6} + (-\sqrt{2} - \sqrt{18})i}{8}$$

$$= -(-\sqrt{2} - 3\sqrt{2})i$$

$$= (\sqrt{2} + 3\sqrt{2})i$$

$$= 4\sqrt{2}i$$

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- [6] 3. Find all of the fourth roots of $16i$. Leave your answers in exponential form, but simplify it.

$$z = 16i = 16e^{i\pi} = 16e^{\frac{i\pi}{2}i} \quad (\text{because } \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = 0 + i(1) = i)$$

$$\text{So } z_k = 16^{\frac{1}{4}} e^{\frac{2\pi k + \pi}{4}i} = (2^4)^{\frac{1}{4}} e^{\frac{(4k+1)\pi}{8}i} \quad \text{where } k=0, 1, 2, 3$$

$$\text{if } k=0, \text{ then } z_0 = 2 e^{\frac{\pi}{8}i}$$

$$\text{if } k=1, \text{ then } z_1 = 2 e^{\frac{5\pi}{8}i}$$

$$\text{if } k=2, \text{ then } z_2 = 2 e^{\frac{9\pi}{8}i}$$

$$\text{if } k=3, \text{ then } z_3 = 2 e^{\frac{13\pi}{8}i}$$

4. Consider the polynomial equation $P(x) = 0$ where $P(x) = 2x^4 + 2x^3 + x^2 + 5x - 10$.

- [3] (a) Use Rational Root Theorem to determine all possible rational roots of $P(x)$.

If $\frac{p}{q}$ is a rational root of $P(x)$, then $p \mid -10$ and $q \mid 2$ so

$$p \in \{\pm 1, \pm 2, \pm 5, \pm 10\} \text{ and } q \in \{\pm 1, \pm 2\}; \text{ therefore}$$

$$\frac{p}{q} \in \{\pm 1, \pm 2, \pm 5, \pm 10, \pm \frac{1}{2}, \pm \frac{5}{2}\}$$

- [3] (b) Let $Q(x) = (x-5)P(x)$. How many positive real roots and negative real roots does $Q(x)$ have? Explain.

$x=5$ is one positive root of $Q(x)$ and since in $P(x)$ sign changes once so $P(x)$ has one positive real root which means $Q(x)$ has 2 positive real roots. For negative real roots of $Q(x)$, since $Q(x) = (x-5)P(x) = 2x^5 - 8x^4 - 9x^3 - 35x + 50$ and $Q(-x) = -2x^5 - 8x^4 + 9x^3 + 35x + 50$ and sign changes once in $Q(-x)$ so the number of negative real roots of $Q(x)$ is exactly 1.

- [3] (c) Use Bounds Theorem to determine an upper bound for the modulus of roots of $P(x)$. Does this eliminate any possible rational roots? Which ones?

$$M = \max\{|12|, |11|, |15|, |-10|\} = 10, \text{ so } |x| < \frac{10}{2} + 1 = 6$$

Therefore ± 10 can be eliminated as possible roots.

- [6] (d) Find all of the solutions to $P(x) = 0$.

$$P(1) = 0 \text{ because } 2+2+1+5-10 = 0$$

$$P(-2) = 0 \text{ because } 2(-2)^4 + 2(-2)^3 + (-2)^2 + 5(-2) - 10 = 32 - 16 + 4 - 10 - 10 = 0$$

so $x-1$ and $x+2$ are factors which means $(x-1)(x+2) = x^2 + x - 2$

is a factor as well and by long division

$$P(x) = (x-1)(x+2)(x^2+x-2) \quad | \quad \begin{array}{r} 2x^2+5 \\ (x^2+x-2) \end{array}$$

$$\text{so } 2x^2+5x-2 \text{ gives } x^2 = -\frac{5}{2} \quad | \quad \begin{array}{r} 2x^4+2x^3+x^2+5x-10 \\ -(2x^4+2x^3-4x^2) \\ \hline 5x^2+5x-10 \end{array}$$

$$\text{which means } x = \pm \sqrt{\frac{5}{2}} \text{ i.e. } \pm \frac{\sqrt{10}}{2} \quad | \quad \begin{array}{r} -(5x^2+5x-10) \\ 0 \end{array}$$

$$\text{Thus } 1, -2, -\frac{\sqrt{10}}{2}, \text{ and } +\frac{\sqrt{10}}{2} \text{ are solutions.}$$

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[8] 5. Let

$$A = \begin{bmatrix} 2 & k \\ 1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ 1 & 2 \\ -1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} -1 & 1 \\ -4 & -3 \end{bmatrix}.$$

Find value(s) of k for which $B^T A - 2C^2 + I = D$.

$$\begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & k \\ 1 & 2 \\ -1 & 0 \end{bmatrix} - 2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 4k+1 \\ 4 & k+4 \end{bmatrix} - 2 \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 8 - 10 + 1 & 4k+1 - 8 + 0 \\ 4 - 8 + 0 & k+4 - 10 + 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & -3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4k-7 \\ -4 & k-5 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & -3 \end{bmatrix}$$

So must $4k-7=1$ and $k-5=-3$ which means

$$k=2.$$

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6. Let $\mathbf{u} = \langle 1, 2, 4 \rangle$, $\mathbf{v} = \langle -1, 0, 1 \rangle$ and $\mathbf{w} = \langle 2, 1, -1 \rangle$.

- [3] (a) Find the angle between the vectors \mathbf{u} and \mathbf{w} .

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{w}} = \langle 1, 2, 4 \rangle \cdot \langle 2, 1, -1 \rangle$$

$$= 2 + 2 - 4$$

$$= 0$$

$$\text{So } \hat{\mathbf{u}} \perp \hat{\mathbf{w}} \text{ that is } \theta = \frac{\pi}{2}$$

- [7] (b) Find a unit vector in the direction of the vector $\mathbf{r} = ((\mathbf{u} + \mathbf{v}) \cdot \mathbf{w})(\mathbf{v} - 2\mathbf{u})$.

$$\hat{\mathbf{u}} + \hat{\mathbf{v}} = \langle 1, 2, 4 \rangle + \langle -1, 0, 1 \rangle = \langle 0, 2, 5 \rangle$$

$$(\hat{\mathbf{u}} + \hat{\mathbf{v}}) \cdot \hat{\mathbf{w}} = \langle 0, 2, 5 \rangle \cdot \langle 2, 1, -1 \rangle = 0 + 2 - 5 = -3$$

$$\hat{\mathbf{v}} - 2\hat{\mathbf{u}} = \langle -1, 0, 1 \rangle - 2 \langle 1, 2, 4 \rangle = \langle -3, -4, -7 \rangle$$

$$\mathbf{r} = ((\hat{\mathbf{u}} + \hat{\mathbf{v}}) \cdot \hat{\mathbf{w}})(\hat{\mathbf{v}} - 2\hat{\mathbf{u}}) = -3 \langle -3, -4, -7 \rangle = 3 \langle 3, 4, 7 \rangle$$

$$\|\mathbf{r}\| = \|3 \langle 3, 4, 7 \rangle\| = \sqrt{3^2 + 4^2 + 7^2} = \sqrt{9 + 16 + 49} = \sqrt{74}$$

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{1}{\sqrt{74}} \langle 3, 4, 7 \rangle = \left\langle \frac{3}{\sqrt{74}}, \frac{4}{\sqrt{74}}, \frac{7}{\sqrt{74}} \right\rangle$$