## MATH 1210 (FALL TERM 2019) SOLUTIONS TO ASSIGNMENT ONE

Q1. Let $P(n)$ denote the statement:

$$
\begin{equation*}
(n+3)+(n+4)+(n+5)+\cdots+(3 n+1)=2(n+1)(2 n-1) \tag{1}
\end{equation*}
$$

(a) Check the statement $P(2)$. That is to say, write down the left-hand side of (1) for $n=2$, and verify that it agrees with the right-hand side.
(b) Check the statement $P(3)$ similarly.
(c) Now use the Principle of Mathematical Induction to prove $P(n)$ for $n \geqslant 1$. The calculations in (a) and (b) should help you in understanding the step

$$
P(k) \Longrightarrow P(k+1) .
$$

Hint: If you continue to have difficulty in spotting the pattern, then write down the statements $P(4), P(5)$ etc and stare at them.

Solution: Throughout, LHS and RHS will mean the left-hand side and right-hand side respectively.
(a) For $n=2$, we have

$$
\mathrm{LHS}=5+6+7=18, \quad \text { RHS }=2.3 .3=18
$$

(b) For $n=3$, we have

$$
\text { LHS }=6+7+8+9+10=40, \quad \text { RHS }=2.4 .5=40
$$

Notice the pattern: From $n=2$ to $n=3$, the starting point of LHS has shifted by 1, and three more terms have been added in the end. The pattern is even clearer if you write down what happens for $n=4$. In that case,

$$
\text { LHS }=7+8+9+10+11+12+13
$$

(c) Now we will begin with the induction. For $n=1$, we have

$$
\mathrm{LHS}=4, \quad \text { RHS }=2.2 .1=4
$$

which are equal. This completes Step A.

Assume the statement $P(k)$ :

$$
\begin{equation*}
(k+3)+(k+4)+(k+5)+\cdots+(3 k+1)=2(k+1)(2 k-1) . \tag{2}
\end{equation*}
$$

The statement to be proved is $P(k+1)$ :

$$
\begin{equation*}
(k+4)+(k+5)+(k+6)+\cdots+(3 k+4)=2(k+2)(2 k+1) . \tag{3}
\end{equation*}
$$

Notice, once again, that the starting point of the LHS in $P(k+1)$ has shifted by 1 compared to $P(k)$, and it has three more terms in the end. The term $k+3$ is missing from the LHS of $P(k+1)$. The trick is to add it and subtract it. Thus

$$
\begin{aligned}
& (k+4)+(k+5)+(k+6)+\cdots+(3 k+4) \\
= & {[(k+3)+(k+4)+(k+5)+\ldots(3 k+1)]+(3 k+2)+(3 k+3)+(3 k+4)-(k+3) } \\
= & 2(k+1)(2 k-1)+(3 k+2)+(3 k+3)+(3 k+4)-(k+3) \\
= & 2(k+1)(2 k-1)+(8 k+6) \\
= & 4 k^{2}+2 k-2+8 k+6=4 k^{2}+10 k+4=2(k+2)(2 k+1),
\end{aligned}
$$

which proves the statement $P(k+1)$. Hence we have completed Step B, and $P(n)$ is proved for $n \geqslant 1$ by the Principle of Mathematical Induction.

Q2. Let $P(n)$ denote the following statement:

$$
4^{n}+6 n-1 \text { is divisible by } 9 .
$$

(a) Verify that the statements $P(3)$ and $P(4)$ are true.
(b) Now use the Principle of Mathematical Induction to prove $P(n)$ for $n \geqslant 1$.

Solution: Let us reformulate the statement $P(n)$ as follows:

$$
\frac{4^{n}+6 n-1}{9} \text { is an integer. }
$$

(a) For $n=3$, we have $\frac{4^{3}+6.3-1}{9}=\frac{81}{9}=9$, which is an integer. For $n=4$, we have $\frac{4^{4}+6.4-1}{9}=\frac{279}{9}=31$, which is also an integer. Hence $P(3)$ and $P(4)$ are true.
(b) For $n=1$, we have $\frac{4^{1}+6.1-1}{9}=\frac{9}{9}=1$ which is an integer. This completes Step A. For Step B, we assume the statement $P(k)$ which says that

$$
\begin{equation*}
\frac{4^{k}+6 k-1}{9}=m \tag{4}
\end{equation*}
$$

is an integer. We want to prove the statement $P(k+1)$, which says that

$$
\begin{equation*}
\frac{4^{k+1}+6(k+1)-1}{9}=\frac{4.4^{k}+6 k+5}{9} \tag{5}
\end{equation*}
$$

is also an integer. The trick is to rewrite $4^{k}$ in terms of $m$. From (4), we get $4^{k}=9 m-6 k+1$. Substitute this into (5) to get

$$
\frac{4(9 m-6 k+1)+6 k+5}{9}=\frac{36 m-18 k+9}{9}=4 m-2 k+1
$$

which is also an integer. Hence Step B is complete, and $P(n)$ is proved for $n \geqslant 1$ by the Principle of Mathematical Induction.

Q3. Find the sum

$$
S=\sum_{m=7}^{23}(m-1)\left(m^{2}+5\right)
$$

Your answer should be a single integer. You will need the summation formulae on page 10 of the course notes. (Answer check: if you do this correctly, then the digits in your final answer should add up to 26.)

Solution: Let us make a change of index so that the lower limit is $=1$. Let $r=m-6$, i.e., $m=r+6$. Then $m=7 \Longrightarrow r=1$, and $m=23 \Longrightarrow r=17$. Hence

$$
S=\sum_{r=1}^{17}(r+5)\left((r+6)^{2}+5\right)=\sum_{r=1}^{17}(r+5)\left(r^{2}+12 r+41\right)=\sum_{r=1}^{17}\left(r^{3}+17 r^{2}+101 r+205\right)
$$

Separate the terms to get

$$
S=\underbrace{\sum_{r=1}^{17} r^{3}}_{A}+17 \underbrace{\sum_{r=1}^{17} r^{2}}_{B}+101 \underbrace{\sum_{r=1}^{17} r}_{C}+17 \times 205 .
$$

Using the formulae on page 10 of the course-notes, we get

$$
A=\left(\frac{17.18}{2}\right)^{2}=23409, \quad B=\frac{17.18 .35}{6}=1785, \quad C=\frac{17.18}{2}=153
$$

Hence

$$
S=23409+17.1785+101.153+17.205=72692
$$

The digits add up to $7+2+6+9+2=26$.

Q4.
(a) Convert the following complex number into Cartesian form:

$$
w=\frac{(\overline{1-i})^{3}}{3+2 i}+\frac{1}{1+1 / i}
$$

(b) Verify that

$$
|w|=\frac{3}{26} \sqrt{130}
$$

If this does not work out, then it means that you have made a mistake in part (a).

Solution: In such questions, it is usually best to simplify the parts separately and then combine. We have $\overline{1-i}=1+i$, and $(1+i)^{3}=1+3 i+3 i^{2}+i^{3}=1+3 i-3-i=-2+2 i$. Moreover, $1 / i=-i$. Hence

$$
w=\frac{-2+2 i}{3+2 i}+\frac{1}{1-i}=\frac{(-2+2 i)(1-i)+(3+2 i)}{(3+2 i)(1-i)}=\frac{4 i+3+2 i}{5-i}=\frac{3+6 i}{5-i} .
$$

Multiply the numerator and the denominator by $\overline{5-i}=5+i$, to get

$$
w=\frac{(3+6 i)(5+i)}{(5-i)(5+i)}=\frac{9+33 i}{26}=\frac{9}{26}+\frac{33 i}{26} .
$$

Now check that

$$
|w|=\sqrt{(9 / 26)^{2}+(33 / 26)^{2}}=\frac{\sqrt{1170}}{26}=\frac{3 \sqrt{130}}{26}
$$

Q5. Find all the complex roots of the following equation:

$$
z^{5}+\left(1-i \frac{\sqrt{3}}{2}\right)=1 / 2 .
$$

Express your answers in exponential form using the principal values of the argument.
Solution: The equation simplifies to

$$
z^{5}=\underbrace{-\frac{1}{2}+i \frac{\sqrt{3}}{2}}_{w} .
$$

We will write $w$ in exponential form. We have ${ }^{1}$

$$
|w|=\sqrt{1 / 4+3 / 4}=1, \quad \arg (w)=\frac{2 \pi}{3}+2 \pi k
$$

where $k$ is an arbitrary integer. Hence

$$
z^{5}=w=e^{\left(\frac{2 \pi}{3}+2 \pi k\right) i},
$$

and then

$$
\left.z=\left[e^{\left(\frac{2 \pi}{3}+2 \pi k\right) i}\right]^{1 / 5}=e^{\left(\frac{2 \pi}{15}\right.}+\frac{2 \pi k}{5}\right) i .
$$

Plug in the successive values $k=0,1,2,3,4$. Then we get $z=e^{i \theta}$, where

$$
\begin{equation*}
\theta=\frac{2 \pi}{15}, \quad \frac{8 \pi}{15}, \quad \frac{14 \pi}{15}, \quad \frac{20 \pi}{15}, \quad \frac{26 \pi}{15} . \tag{6}
\end{equation*}
$$

But we want $-\pi<\arg (z) \leqslant \pi$, and hence we need to subtract $2 \pi$ from the last two values in (6). Hence, finally we get the five roots

$$
z=e^{i \theta}, \quad \text { where } \quad \theta=\frac{2 \pi}{15}, \frac{8 \pi}{15}, \frac{14 \pi}{15}, \quad-\frac{10 \pi}{15}, \quad-\frac{4 \pi}{15} .
$$

[^0]Q6. There are exactly two complex numbers $z$ which simultaneously satisfy both of the following equations:

$$
\begin{equation*}
|z+1+3 i|=\sqrt{34}, \quad(1-3 i) z+(1+3 i) \bar{z}=4 \tag{7}
\end{equation*}
$$

Find them. Hint: substitute $z=x+i y$ into the equations and solve.

## Solution:

If we substitute $z=x+i y$, then the first equation becomes $|(x+1)+i(y+3)|=\sqrt{34}$, i.e.,

$$
\begin{equation*}
(x+1)^{2}+(y+3)^{2}=34 \tag{8}
\end{equation*}
$$

Notice that this is a circle in the $\mathrm{X}-\mathrm{Y}$ plane with radius $\sqrt{34}$ and centre $(-1,-3)$. Now make the same substitution in the second equation in (7). Then we get $(1-3 i)(x+i y)+(1+3 i)(x-i y)=(x-3 i x+i y+3 y)+(x+3 i x-i y+3 y)=2 x+6 y$.

From $2 x+6 y=4$, we get

$$
\begin{equation*}
x+3 y=2 \tag{9}
\end{equation*}
$$

This is a line in the $\mathrm{X}-\mathrm{Y}$ plane. The geometric picture is that the line and the circle intersect in two points, which we want to find.


From the equation of the line, let $x=2-3 y$. Substitute this into the equation of the circle to get

$$
(3-3 y)^{2}+(y+3)^{2}=\left(9 y^{2}-18 y+9\right)+\left(y^{2}+6 y+9\right)=10 y^{2}-12 y+18=34
$$

i.e., $10 y^{2}-12 y-16=0$. The roots are $y=2,-4 / 5$. If we substitute these $y$-values into the equation of the line, then we get $x=-4,22 / 5$ respectively. Hence the two $z$-values are

$$
-4+2 i, \quad \text { and } \quad \frac{22}{5}-\frac{4 i}{5}
$$


[^0]:    ${ }^{1}$ Note that when the points are $(x, y)=\left( \pm \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ or $\left( \pm \frac{\sqrt{3}}{2}, \pm \frac{1}{2}\right)$, you should know their arguments by heart.

