UNIVERSITY OF MANITOBA
COURSE: MATH 1210
DATE & TIME: 31 October, 2019, 5:45 pm - 7 pm
DURATION: 75 minutes EXAMINER: Jaydeep Chipalkatti and Donald Trim

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Solutions to the Midterm

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[8] 1. Use the Principle of Mathematical Induction to prove the equality

$$\frac{1}{1.\,2}+\frac{1}{2.\,3}+\frac{1}{3.\,4}+\cdots+\frac{1}{n\,(n+1)}=\frac{n}{n+1}$$

for all $n \geqslant 1$.

Solution: Let $P(n)$ denote the given statement.

Step A: We verify $P(1)$. The LHS (left-hand side) of $P(1)$ is equal to $\frac{1}{1.2}=\frac{1}{2}$. The RHS (right-hand side) is also $\frac{1}{1+1}=\frac{1}{2}$. Hence $P(1)$ is true.

Step B: Now assume the statement $P(k)$, which is:

$$\frac{1}{1.\,2}+\frac{1}{2.\,3}+\frac{1}{3.\,4}+\cdots+\frac{1}{k\,(k+1)}=\frac{k}{k+1}.$$

The statement to be proved is $P(k+1)$, which is:

$$\frac{1}{1.\,2}+\frac{1}{2.\,3}+\frac{1}{3.\,4}+\cdots+\frac{1}{(k+1)\,(k+2)}=\frac{k+1}{k+2}.$$

Then we have

$$\begin{aligned}&\frac{1}{1.\,2}+\frac{1}{2.\,3}+\frac{1}{3.\,4}+\cdots+\frac{1}{k\,(k+1)}+\frac{1}{(k+1)\,(k+2)}\\ =&\frac{k}{k+1}+\frac{1}{(k+1)\,(k+2)}=\frac{k(k+2)+1}{(k+1)\,(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}=\frac{k+1}{k+2}.\end{aligned}$$

We have proved that $P(k) \implies P(k+1)$, which completes Step B. Hence $P(n)$ is proved for all $n \geqslant 1$, by the Principle of Mathematical Induction.

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[4] 2. Write the sum

$$\frac{3}{2^7} - \frac{4}{2^8} + \frac{5}{2^9} - \frac{6}{2^{10}} + \cdots + \frac{11}{2^{15}}$$

in Σ-notation. Your index of summation should start from 1.

Solution:

The sum can be written as

$$\sum_{i=1}^{9} (-1)^{i+1}\, \frac{i+2}{2^{i+6}}.$$

Of course, you can use any letter in place of i.

[7] 3. Convert the complex number

$$\frac{\overline{1+i}}{\left(\sqrt{2}+e^{\frac{3\pi\, i}{4}}\right)^6}$$

into Cartesian form.

Solution:

Let z denote this number. Such problems are often easier if you simplify the numerator and the denominator separately. The numerator is $\overline{1+i} = 1 - i$.

As to the denominator, notice that

$$e^{\frac{3\pi\, i}{4}} = \cos\frac{3\pi}{4} + i\,\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}.$$

Hence

$$\sqrt{2} + e^{\frac{3\pi\, i}{4}} = \sqrt{2} - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{2-1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = e^{\frac{\pi\, i}{4}},$$

and then

$$\text{denominator} = (\sqrt{2} + e^{\frac{3\pi\, i}{4}})^6 = (e^{\frac{\pi\, i}{4}})^6 = e^{\frac{6\pi\, i}{4}} = e^{\frac{3\pi\, i}{2}} = \cos\frac{3\,\pi}{2} + i\sin\frac{3\pi}{2} = 0 - i = -i.$$

Hence

$$z = \frac{1-i}{-i} = \frac{(1-i)\,i}{-i.i} = \frac{(1-i)\,i}{-i^2} = \frac{(1-i)\,i}{1} = 1 + i.$$

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[7] 4. Consider the polynomial

$$f(x) = x^3 + a\,x^2 + b\,x + c,$$

where a, b, c are real numbers. It is given to you that $3 - i$ and 2 are roots (= zeros) of $f(x)$. Find the values of a, b, c.

Solution:

Since a, b, c are real numbers, $f(x)$ is a polynomial with real coefficients. Since $3 - i$ is a root, its conjugate $3 + i$ is also a root. (This follows from Theorem 3.5 in Section 3.1 of our course notes.) Since $\deg f(x) = 3$, we know that

$$3 - i, \quad 3 + i, \quad 2$$

are all the roots of $f(x)$, and hence

$$f(x) = (x - (3 - i))\,(x - (3 + i))\,(x - 2).$$

Now it is just a matter of multiplying this out carefully. We have

$$(x - (3 - i))\,(x - (3 + i)) = x^2 - 6x + 10,$$

and then

$$f(x) = (x^2 - 6x + 10)\,(x - 2) = x^3 - 8\,x^2 + 22\,x - 20.$$

Hence

$$a = -8, \quad b = 22, \quad \text{and} \quad c = -20.$$

[5] 5. Consider the matrix $A = \begin{bmatrix} 3 & -1 \\ 2 & -5 \end{bmatrix}$, and let I_2 denote the 2×2 identity matrix. Now find the matrix

$$A^2 + 2\,A - 13\,I_2.$$

Solution: We have

$$A^2 = \begin{bmatrix} 3 & -1 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ -4 & 23 \end{bmatrix},$$

and hence

$$A^2 + 2\,A - 13\,I_2 = \begin{bmatrix} 7 & 2 \\ -4 & 23 \end{bmatrix} + 2 \begin{bmatrix} 3 & -1 \\ 2 & -5 \end{bmatrix} - 13 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

[4] 6. Let I_n denote the $n \times n$ identity matrix. Let A and B be matrices such that the expression

$$B^T I_3 A - 5 I_4$$

is defined. Find the sizes of A and B.

Solution:

Let us suppose that A is $a \times b$, and B is $c \times d$. Then B^T is $d \times c$. Since $B^T I_3$ is defined, we must have $c = 3$. Since $I_3 A$ is defined, $a = 3$. Now $B^T I_3 A$ is $d \times b$. But we know that it must be 4×4, since its addition with $-5 I_4$ is defined. Hence $d = 4, b = 4$.

In conclusion, A and B are both 3×4.

[6] 7. Consider the vectors

$$\mathbf{u} = \langle 4, a, 7 \rangle \quad \text{and} \quad \mathbf{v} = \langle a, a-1, -4 \rangle,$$

in $\mathbb{R}^3$. It is given to you that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular (i.e., orthogonal). Find all possible values of a.

Solution:

Since $\mathbf{u}, \mathbf{v}$ are perpendicular, we have $\mathbf{u}.\mathbf{v} = 0$. Thus

$$0 = 4\,a + a\,(a-1) - 28 = a^2 + 3\,a - 28 = (a+7)\,(a-4).$$

Hence $a = -7$ or 4.

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[9] 8. Consider the two lines

$$L_1: \quad x = 2 - t, \quad y = 3 + t, \quad z = -4 - 6\,t,$$

and

$$L_2: \quad x = s, \quad y = 2 - s, \quad z = 3 + 6\,s$$

in $\mathbb{R}^3$.

(a) Show that L_1 and L_2 are parallel. (That is to say, give a complete mathematical reasoning.)

(b) Now find the equation of the unique plane which contains both L_1 and L_2.

Solution: Using the parametric equations above, we see that the direction vectors for L_1 and L_2 are respectively

$$\langle -1, 1, -6 \rangle, \quad \text{and} \quad \langle 1, -1, 6 \rangle.$$

Since they are negatives of each other, we know that either L_1, L_2 are parallel or they are the same line. We need to show that they are distinct lines. Take the point $P = (2, 3, -4)$ on L_1. If it were on L_2, then the equations

$$2 = s, \quad 3 = 2 - s, \quad -4 = 3 + 6s$$

would have been consistent, but they are not. Hence $(2, 3, -4)$ is not on L_2 and the lines are distinct. It follows that they are parallel.

Let Π denote the plane containing L_1 and L_2. In order to find the normal $\mathbf{n}$ to Π, take any point Q on L_2. For instance, by letting $s = 0$, we can take $Q = (0, 2, 3)$. Then the vector $PQ = \langle -2, -1, 7 \rangle$ is in Π. Now we can take

$$\mathbf{n} = \text{direction vector of } L_1 \times PQ = \langle -1, 1, -6 \rangle \times \langle -2, -1, 7 \rangle$$

$$\begin{vmatrix} i & j & k \\ -1 & 1 & -6 \\ -2 & -1 & 7 \end{vmatrix} = \begin{vmatrix} 1 & -6 \\ -1 & 7 \end{vmatrix} i - \begin{vmatrix} -1 & -6 \\ -2 & 7 \end{vmatrix} j + \begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix} k$$

$$= i + 19\,j + 3\,k = \langle 1, 19, 3 \rangle.$$

Hence the equation of Π must be of the form $x + 19\,y + 3\,z = k$ for some constant k. Now substitute the coordinates of $P = (2, 3, -4)$ to get $k = 2 + 19.3 + 3.(-4) = 47$. Hence the equation of Π is

$$x + 19\,y + 3\,z = 47.$$