Solutions to Assignment-1

Q1. This question is about the uniqueness of the reduced row-echelon form (RREF). Consider the matrix

$$M = \begin{bmatrix} 3 & -12 & 1 & 14 \\ 1 & -4 & 0 & 3 \\ 7 & -28 & 3 & 36 \end{bmatrix}.$$

1. Begin with M. Start with the row operation $R_1/3$ followed by $R_2 - R_1$ and go on to reduce M to its RREF.
2. Begin with M again. Now start with the row operation $R_1 \leftrightarrow R_2$ followed by $R_2 - 3\,R_1$ and go on to reduce M to its RREF.

Your final answers in (1) and (2) must be *exactly* the same.

Answer: By either route, the final RREF comes out to be

$$\begin{bmatrix} 1 & -4 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We will work out the details during the lecture.

Q2. Consider the matrices

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 7 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -8 & 5 \\ 6 & 0 & 1 \\ -4 & 2 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & -9 \\ 8 & -1 \\ 3 & 4 \end{bmatrix}.$$

- Calculate $A\,B$.
- Now multiply on the right by C to calculate $(A\,B)\,C$.
- Calculate $B\,C$.
- Now multiply on the left by A to calculate $A\,(B\,C)$.

You must get $(A\,B)\,C = A\,(B\,C)$.

Answer: The matrices come out to be

$$A\,B = \begin{bmatrix} -20 & -14 & 6 \\ 54 & -36 & 29 \end{bmatrix}, \quad (A\,B)\,C = \begin{bmatrix} -194 & 218 \\ 69 & -334 \end{bmatrix},$$

and

$$B\,C = \begin{bmatrix} -44 & 19 \\ 33 & -50 \\ -7 & 30 \end{bmatrix}, \quad A\,(B\,C) = \begin{bmatrix} -194 & 218 \\ 69 & -334 \end{bmatrix}.$$

Hence

$$(A\,B)\,C = A\,(B\,C).$$

Q3. Consider the following system of equations:

$$2x + y + z = 1, \quad x - 3y + 5z = 10, \quad -x + y - 6z = -1.$$

1. Solve the system using Gauss-Jordan elimination (i.e., by converting the augmented matrix to RREF).
2. Now solve the same system using Cramer's rule.

Of course, your solutions in (1) and (2) must agree with each other.

Answer: The augmented matrix is

$$A = \left[\begin{array}{rrr|r} 2 & 1 & 1 & 1 \\ 1 & -3 & 5 & 10 \\ -1 & 1 & -6 & -1 \end{array}\right].$$

Its RREF comes out to be

$$R = \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

(We will work out the details during the lecture.) Hence the solution is

$$x = 3, \quad y = -4, \quad z = -1.$$

Cramer's rule: The matrix form of the equation is

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 1 & -3 & 5 \\ -1 & 1 & -6 \end{bmatrix}}_{C} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -1 \end{bmatrix}.$$

Check that

$$\det C = 25.$$

Now replace the first column of C by $\begin{bmatrix} 1 \\ 10 \\ -1 \end{bmatrix}$. This gives the matrix

$$C_x = \begin{bmatrix} 1 & 1 & 1 \\ 10 & -3 & 5 \\ -1 & 1 & -6 \end{bmatrix}.$$

Now check that $\det C_x = 75$, and hence

$$x = \frac{\det\, C_x}{\det\, C} = \frac{75}{25} = 3.$$

Similarly, we get $\det\, C_y = -100$, $\det\, C_z = -25$, and hence

$$y = \frac{\det\, C_y}{\det\, C} = -\frac{100}{25} = -4, \qquad z = \frac{\det\, C_z}{\det\, C} = -\frac{25}{25} = -1.$$

Q4. Consider the matrix

$$M = \begin{bmatrix} 3 & -7 & 5 \\ -4 & 11 & 9 \\ 2 & -8 & 10 \end{bmatrix}.$$

- Find $\det(M)$ by cofactor expansion along column 2.
- Find $\det(M)$ by cofactor expansion along row 3.

Of course, both answers must agree with each other. Now calculate the same determinant by using row-operations to convert it into upper triangular form.

Answer: Expanding by column 2, the formula is

$$\det M = a_{12}\, C_{12} + a_{22}\, C_{22} + a_{32}\, C_{32}.$$

Now for example,

$$C_{12} = -M_{12} = -\begin{vmatrix} -4 & 9 \\ 2 & 10 \end{vmatrix} = -(-58) = 58.$$

Similarly, $C_{22} = 20, C_{33} = -47$. Hence

$$\begin{aligned} \det M &= a_{12}\, C_{12} + a_{22}\, C_{22} + a_{32}\, C_{32} \\ &= (-7) \times 58 + 11 \times 20 + (-8) \times (-47) = 190. \end{aligned}$$

Expanding by row 3, the formula is

$$\det M = a_{31}\, C_{31} + a_{32}\, C_{32} + a_{33}\, C_{33}.$$

Now for example, $C_{31} = M_{31} = \begin{vmatrix} -7 & 5 \\ 11 & 9 \end{vmatrix} = -118$. Similarly, $C_{32} = -47, C_{33} = 5$. Hence

$$\begin{aligned} \det M &= a_{31}\, C_{31} + a_{32}\, C_{32} + a_{33}\, C_{33} \\ &= 2 \times (-118) + (-8) \times (-47) + 10 \times 5 = 190. \end{aligned}$$

We get the same answer by the row-operations method; we will check this during the lecture.

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