Solutions to Assignment 2

Q1. Let

$$A = \begin{bmatrix} 7 & -2 & -9 \\ -5 & 1 & 5 \\ 3 & -1 & -4 \end{bmatrix}.$$

(a) Find A^{-1} using the adjoint formula.

(b) Find A^{-1} using the row-operations method.

Answer: Check that $\det(A) = -1$.

Now we need to calculate all the minors. For example,

$$M_{12} = \begin{vmatrix} -5 & 5 \\ 3 & -4 \end{vmatrix} = 5.$$

This way, we get

$$\mathrm{Min}(A) = \begin{bmatrix} 1 & 5 & 2 \\ -1 & -1 & -1 \\ -1 & -10 & -3 \end{bmatrix}.$$

Now change the signs according to $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$. Then

$$\mathrm{Cof}(A) = \begin{bmatrix} 1 & -5 & 2 \\ 1 & -1 & 1 \\ -1 & 10 & -3 \end{bmatrix},$$

and

$$\mathrm{Adj}(A) = \mathrm{Cof}(A)^T = \begin{bmatrix} 1 & 1 & -1 \\ -5 & -1 & 10 \\ 2 & 1 & -3 \end{bmatrix}.$$

Hence

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{Adj}(A) = \begin{bmatrix} -1 & -1 & 1 \\ 5 & 1 & -10 \\ -2 & -1 & 3 \end{bmatrix},$$

For part (b), we start with the double matrix $[A|I]$ and convert to $[I|A^{-1}]$ using row-operations. We will work this out in class.

Q2. Let

$$w = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}.$$

Find all the fifth roots of w. Your answers must be in exponential form using the principal values of the argument. Do not use a calculator to solve this question.

Answer: First, convert w to exponential form. Now

$$|w| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1.$$

The pair of numbers $(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$ should tell you that this is a standard angle (i.e., one of $30^\circ, 45^\circ, 60^\circ$ in any quadrant), and you should simply know the sines and cosines of such angles by heart.

Hence, you should be able to guess that $\arg(w) = \frac{2\pi}{3} + 2n\pi$, where n is an arbitrary integer. Thus

$$w = e^{i\left(\frac{2\pi}{3} + 2n\pi\right)}.$$

Now let $z = re^{i\theta}$ be an fifth root of w, i.e., $z^5 = w$. This turns into

$$r^5 e^{5i\theta} = 1\, e^{i(\frac{2\pi}{3} + 2n\pi)},$$

hence $r^5 = 1 \implies r = 1$.

Now

$$5\theta = \frac{2\pi}{3} + 2n\pi \implies \theta = \frac{2\pi}{15} + \frac{2n\pi}{5} = \frac{(2+6n)\pi}{15}.$$

To get 5 distinct values of θ, substitute $n = 0, 1, 2, 3, 4$. This gives

$$\theta = \frac{2\pi}{15}, \quad \frac{8\pi}{15}, \quad \frac{14\pi}{15}, \quad \frac{20\pi}{15}, \quad \frac{26\pi}{15}.$$

However we want principal values of the argument, which means $-\pi < \arg(z) \leqslant \pi$. Hence we subtract 2π from the last two values, giving

$$\theta = \frac{2\pi}{15}, \quad \frac{8\pi}{15}, \quad \frac{14\pi}{15}, \quad -\frac{10\pi}{15} = -\frac{2\pi}{3}, \quad -\frac{4\pi}{15}. \tag{1}$$

Hence the answer is $z = e^{i\theta}$, where θ is any of the values from (1).

Alternately you can use De Moivre's formula, but the final answers should come out to be the same. In fact, De Moivre's formula is essentially a special case of Euler's theorem.

Q3. Let

$$w = 5 - 7\,i.$$

(a) Find the principal value of $arg(w)$ to four decimal places. For example, a number such as 1.2345 is written to four decimal places.

(b) Now find all the cube-roots of w. Your answers must be in Cartesian form, accurate to four decimal places.

Answer: This is similar to Q2, but everything is numerical. Note that

$$|w| = \sqrt{5^2 + (-7)^2} = \sqrt{74}.$$

Let $\alpha = \arg(w)$ be the principal value. Then

$$\sin\alpha = -\frac{7}{\sqrt{74}},$$

and hence

$$\alpha = \sin^{-1}(-\frac{7}{\sqrt{74}}) \simeq -0.9505468408 \simeq -0.9505$$

accurate to four decimal places. Thus

$$w = \sqrt{74}\, e^{i(\alpha + 2n\pi)}$$

where n is an arbitrary integer.

Now let $z = r\,e^{i\theta}$ be a cube-root of w. Then the equation $z^3 = w$ becomes

$$r^3 e^{3i\theta} = \sqrt{74}\, e^{i(\alpha+2n\pi)}.$$

First,

$$r^3 = \sqrt{74} \implies r = \sqrt[6]{74}.$$

Secondly,

$$3\,\theta = \alpha + 2n\pi \implies \theta = \frac{\alpha + 2n\pi}{3}.$$

Now substitute $n = 0, 1, 2$. Thus we have

$$z = \sqrt[6]{74}\, e^{i\theta} = \sqrt[6]{74}\, (\cos\theta + i\, \sin\theta),$$

where

$$\theta = \frac{\alpha}{3}, \quad \frac{\alpha + 2\pi}{3}, \quad \frac{\alpha + 4\pi}{3}.$$

Now use a calculator to evaluate these numbers. We get the following three cube-roots:

$$1.9470 - 0.6384\, i, \quad -0.4206 + 2.0053\, i, \quad -1.5264 - 1.3669\, i$$

accurate to four decimal places.

Q4. Consider the matrix

$$A = \begin{bmatrix} 1 & x & 0 \\ -2 & 4 & 9 \\ 3 & y & -1 \end{bmatrix}.$$

It is given to you that

(a) The $(2,3)$ entry of A^{-1} is $\frac{3}{22}$.

(b) The $(3,3)$ entry of A^{-1} is $\frac{1}{11}$.

Find the values of x, y.

Answer: We have seen the formula

$$\text{The } (i,j) \text{ th entry of } A^{-1} = \frac{C_{ji}}{\det(A)}, \tag{2}$$

where C_{ji} is the (j,i)-th cofactor of A.

This is the procedure for solving the question:

- For $(i,j) = (2,3)$, we calculate the right-hand side of the formula as given, and make it equal to the $\frac{3}{22}$ (which is part of the data). This will give us an equation for x, y.
- Do the same for $(i,j) = (3,3)$; this will give us another equation for x, y.
- Now solve the two equations to find x and y.

It is just a matter of carrying this through. First, check that

$$\det A = 25\,x - 9\,y - 4.$$

This can be done by any method; for example, cofactor expansion or basket-weave.

Now let $(i, j) = (2, 3)$. Then

$$C_{32} = -M_{32} = -\begin{vmatrix} 1 & 0 \\ -2 & 9 \end{vmatrix} = -9.$$

Hence (2) becomes

$$\frac{3}{22} = \frac{-9}{25\,x - 9\,y - 4}.$$

After cross-multiplying, this becomes the equation

$$-75\,x + 27\,y = 186. \tag{3}$$

Now let $(i, j) = (3, 3)$. Then

$$C_{33} = M_{33} = \begin{vmatrix} 1 & x \\ -2 & 4 \end{vmatrix} = 4 + 2x.$$

Hence (2) becomes

$$\frac{1}{11} = \frac{4 + 2x}{25\,x - 9\,y - 4}.$$

After cross-multiplying, this becomes the equation

$$3\,x - 9\,y = 48. \tag{4}$$

Now solve the system of linear equations (3) and (4). This can be done via several methods; for example, Gauss-Jordan or Cramer's rule. The answer comes out to be

$$x = -5, \quad y = -7.$$