Solutions to Assignment 3

Q1. Consider the polynomial

$$f(x) = 6x^5 + 25x^4 - 5x^3 - 110x^2 - 121x - 35.$$

Note that $f(x)$ has 5 roots.

(a) Use the rational roots theorem to list all possible rational roots of $f(x)$.

(b) Now use the Bounds Theorem to find the upper bound Q. Then eliminate some of the possible roots from (a).

(c) Now check by possible substitution which of the possible rational roots are actual roots.

(d) Now find all the remaining roots of $f(x)$.

Suppose that p/q is a rational root. Then

$$p \text{ divides } 35 \implies p = \pm 1, \pm 5, \pm 7, \pm 35$$
$$q \text{ divides } 6 \implies q = \pm 1, \pm 2, \pm 3, \pm 6.$$

Hence the possible rational roots are $\pm p/q$ where

$$p \in \{1, 5, 7, 35\}, \quad q \in \{1, 2, 3, 6\}.$$

Now we have to substitute these into $f(x)$ and check if the result is zero. There are 32 possibilities in all.

Using the Bounds Theorem, we can reduce the work a little. Notice that $L = 6$, and

$$M = \max\{25, 5, 110, 121, 35\} = 121.$$

Hence

$$Q = \frac{M}{L} + 1 = \frac{121}{6} + 1 \simeq 21.2.$$

Since $|r| < Q$ for any root r, we know that ± 35 cannot be roots, but that still leaves 30 possibilities. As mentioned in the question, you need to show the substitution in detail for two of them, and you can check the rest by some sort of computer program.

The actual rational roots turn out to be:

$$-\frac{5}{3}, \quad -\frac{1}{2}, \quad -1.$$

This tells us that $f(x)$ has linear factors

$$(3x+5), \quad (2x+1), \quad (x+1).$$

Hence $f(x)$ is divisible by

$$(3x+5)(2x+1)(x+1) = 6\,x^3 + 19\,x^2 + 18\,x + 5.$$

Now do a long division:

$$
\begin{array}{r}
x^2 + x - 7 \\
6x^3+19x^2+18x+5 \,\big)\, \overline{6x^5+25x^4-5x^3-110x^2-121x-35} \\
\underline{-6x^5-19x^4-18x^3-5x^2} \\
6x^4-23x^3-115x^2-121x \\
\underline{-6x^4-19x^3-18x^2-5x} \\
-42x^3-133x^2-126x-35 \\
\underline{42x^3+133x^2+126x+35} \\
0
\end{array}
$$

The quotient is:

$$x^2 + x - 7.$$

It doesn't factor, hence we use the quadratic formula:

$$\begin{aligned} x^2 + x - 7 = 0 \implies x &= \frac{-1 \pm \sqrt{1^2 - 4 \times (-7)}}{2} \\ &= \frac{-1 \pm \sqrt{29}}{2} \end{aligned}$$

which are the remaining two roots.

Q2. Consider the polynomial

$$f(x) = x^5 - 10\,x^4 + 27\,x^3 + 24\,x^2 - 98\,x + 116.$$

It is given to you that $1 - i$ and $5 + 2\,i$ are two of the roots of $f(x)$. Find all the remaining roots.

Since all coefficients of $f(x)$ are real, the Conjugate Root Theorem applies. We are given that

$$z = 1 - i, \qquad w = 5 + 2\,i$$

are roots of $f(x)$, hence

$$\bar{z} = 1 + i, \qquad \overline{w} = 5 - 2\,i$$

are also roots.

Hence

$$g(x) = \underbrace{(x - z)\,(x - \overline{z})}_{g_1(x)}\,\underbrace{(x - w)\,(x - \overline{w})}_{g_2(x)}$$

is a factor of $f(x)$.

Now

$$g_1(x) = x^2 - (z + \overline{z})\,x + z\,\overline{z} = x^2 - 2x + 2,$$

$$g_2(x) = x^2 - (w + \overline{w})\,x + w\,\overline{w} = x^2 - 10\,x + 29.$$

At this point, there are two ways to divide $f(x)$ by $g(x)$:

- First divide by $g_1(x)$ and then divide the quotient by $g_2(x)$, or
- Divide $f(x)$ once and for all by the product

$$g(x) = x^4 - 12\,x^3 + 51\,x^2 - 78\,x + 58.$$

The long division is shown on the next page, but the first option would have been fine too.

$$
\begin{array}{r|l}
\multicolumn{2}{r}{x \quad + 2} \\
\cline{2-2}
x^4 - 12x^3 + 51x^2 - 78x + 58 & x^5 - 10x^4 + 27x^3 + 24x^2 - 98x + 116 \\
 & -x^5 + 12x^4 - 51x^3 + 78x^2 - 58x \\
\cline{2-2}
 & 2x^4 - 24x^3 + 102x^2 - 156x + 116 \\
 & -2x^4 + 24x^3 - 102x^2 + 156x - 116 \\
\cline{2-2}
 & 0
\end{array}
$$

Hence $f(x) = g(x)\,(x+2)$, which implies that the remaining root is -2. This is the complete list of five roots:

$$1 - i, \quad 1 + i, \quad 5 - 2\,i, \quad 5 + 2\,i, \quad -2.$$

Q3. Consider the statement $\mathbb{P}(n)$:

$$2^2 + 7^2 + 12^2 + 17^2 + \cdots + (5n-3)^2 = \frac{25}{3}\,n^3 - \frac{5}{2}\,n^2 - \frac{11}{6}\,n.$$

(a) Check that the statement $\mathbb{P}(4)$ is true by calculating the LHS and RHS. A calculator is permitted, but you should show your calculations in reasonable detail.

(b) Now prove the statement $\mathbb{P}(n)$ for $n \geqslant 1$, by using the Principle of Mathematical Induction.

Let us check $\mathbb{P}(4)$. The last term on the LHS is $5 \times 4 - 3 = 17$, hence the entire LHS is

$$2^2 + 7^2 + 12^2 + 17^2 = 486.$$

The RHS is

$$\frac{25}{3} \times 4^3 - \frac{5}{2} \times 4^2 - \frac{11}{6} \times 4 = 486.$$

Since both sides are equal, $\mathbb{P}(4)$ is true.

Step A: Let us check the case $n = 1$. The LHS of $\mathbb{P}(1)$ is $2^2 = 4$, and the RHS is

$$\frac{25}{3} - \frac{5}{2} - \frac{11}{6} = \frac{50 - 15 - 11}{6} = \frac{24}{6} = 4.$$

Hence $\mathbb{P}(1)$ is true.

Step B: Now assume

$$\mathbb{P}(k): \quad 2^2+7^2+12^2+17^2+\cdots+(5k-3)^2 = \frac{25}{3}k^3 - \frac{5}{2}k^2 - \frac{11}{6}k.$$

We have to prove the statement

$$\begin{aligned}\mathbb{P}(k+1): \quad & 2^2+7^2+12^2+17^2+\cdots+(5k-3)^2+(5k+2)^2 \\ &= \frac{25}{3}(k+1)^3 - \frac{5}{2}(k+1)^2 - \frac{11}{6}(k+1) \\ &= \frac{25}{3}(k^3+3k^2+3k+1) - \frac{5}{2}(k^2+2k+1) - \frac{11}{6}(k+1) \\ &= \frac{25\,k^3}{3} + \frac{45\,k^2}{2} + \frac{109\,k}{6} + 4.\end{aligned}$$

By assumption,

$$\text{LHS of } \mathbb{P}(k) = \text{RHS of } \mathbb{P}(k).$$

Now we have the sequence

$$\begin{aligned}
&\text{LHS of } \mathbb{P}(k+1)\\
&= \text{LHS of } \mathbb{P}(k) + (5k+2)^2\\
&= \text{RHS of } \mathbb{P}(k) + 25k^2 + 20k + 4\\
&= \frac{25}{3}k^3 - \frac{5}{2}k^2 - \frac{11}{6}k + 25k^2 + 20k + 4\\
&= \frac{25\,k^3}{3} + \frac{45\,k^2}{2} + \frac{109\,k}{6} + 4\\
&= \text{RHS of } \mathbb{P}(k+1).
\end{aligned}$$

Hence $\mathbb{P}(k+1)$ is proved, and we have established $\mathbb{P}(n)$ for all $n \geqslant 1$ by PMI.

Q4. Consider the statement

$$\mathbb{P}(n) : \quad 5 \text{ divides } n\left(n^4 + 4\right).$$

(a) Check that the statements $\mathbb{P}(3)$ and $\mathbb{P}(7)$ are true.

(b) Now prove the statement $\mathbb{P}(n)$ for $n \geqslant 1$, by using the Principle of Mathematical Induction. You will need the binomial theorem.

For $n = 3$, we have the statement

$$\mathbb{P}(3): \quad 5 \text{ divides } 3 \times \left(3^4 + 4\right) = 255,$$

which is certainly true, since $\frac{255}{5} = 51$ is an integer.

For $n = 7$, we have the statement

$$\mathbb{P}(7): \quad 5 \text{ divides } 7 \times \left(7^4 + 4\right) = 16835,$$

which is also true. Notice that you don't actually need to divide; if an integer ends in 0 or 5 then it is divisible by 5.

Step A is easy. For $n = 1$, we have the statement

$$\mathbb{P}(1): \quad 5 \text{ divides } 1 \times (1^4 + 4) = 5,$$

which is certainly true.

Step B: Now assume

$$\mathbb{P}(k): \quad 5 \text{ divides } k\,(k^4 + 4),$$

which implies that

$$\frac{k\,(k^4+4)}{5} = m$$

is an integer. Hence

$$k^5 + 4\,k = 5m.$$

We want to prove

$$\mathbb{P}(k+1): \quad 5 \text{ divides } (k+1)\,((k+1)^4 + 4).$$

Now we have

$$\begin{aligned}
\frac{(k+1)\left((k+1)^4+4\right)}{5} &= \frac{(k+1)(k^4+4k^3+6k^2+4k+1)+4}{5} \\
&= \frac{k^5+5\,k^4+10\,k^3+10\,k^2+9\,k+5}{5} \\
&= \frac{(k^5+4k)+(5\,k^4+10\,k^3+10\,k^2+5\,k+5)}{5} \\
&= \frac{5m+5\,k^4+10\,k^3+10\,k^2+5\,k+5}{5} \\
&= m+k^4+2k^3+2k^2+k+1,
\end{aligned}$$

which is an integer. Hence $\mathbb{P}(k+1)$ is proved and we have proved $\mathbb{P}(n)$ for all $n \geqslant 1$ by PMI.