# Final Examination 

Due: Saturday December 19, 2020 10:00 AM (Central Standard Time)

## Assignment description

(1) The test is open-book and open notes. You can also look up any text or online source you want. A calculator is permitted.
(2) You must show your work in detail in order to get marks.
(3) Note that, in many cases, you can check your answer (for example, Q5, Q6 etc). Try to do this wherever possible. This is optional, and you won't get any extra marks for checking your answer, but it will be reassuring to you.
(4) You cannot seek help from any other person while writing the test.
(5) You may communicate with the instructor by email in order to seek clarification about a question if necessary. However, before writing to the instructor, read the question carefully and also look up all the online notes. I will not respond if I think that you are merely fishing for hints.
(6) You must submit your answers by 10 am on Saturday, December 19th. You will lose all marks if you submit late. No honesty declaration is necessary.

## Submit your assignment

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

## Q1 (7 points)

Consider the summation:

$$
S=3^{2}+8^{2}+13^{2}+\cdots+(5 n-2)^{2}
$$

(a) Express $S$ in $\Sigma$-notation using the index $i$. The lower limit of your summation must be $i=1$.
(b) Now apply the Bernoulli formulae and find a closed expression $E(n)$ for $S$. Your expression must look like

$$
E(n)=a n^{3}+b n^{2}+c n+d
$$

where $a, b, c, d$ are some rational numbers.
(c) Now calculate $S$ and $E(n)$ for $n=5$. They must come out to be equal. If not, you have made a mistake somewhere.

Note that this question has nothing to do with induction.

## Q2 (9 points)

Consider the statement

$$
\mathbb{P}(n): \quad 4^{2}+11^{2}+18^{2}+\cdots+(7 n-3)^{2}=\frac{49 n^{3}}{3}+\frac{7 n^{2}}{2}-\frac{23 n}{6}
$$

(a) Check that $\mathbb{P}(5)$ is true.
(b) Prove $\mathbb{P}(n)$ for $n \geqslant 1$ by the Principle of Mathematical Induction.

Note that this question has nothing to do with the Bernoulli formulae.

## Q3 (6 points)

Consider the polynomial equation $f(x)=0$, where

$$
f(x)=3 x^{6}-2 x^{5}+a x^{4}+b x^{3}-7 x^{2}+8 x-1
$$

where $a$ and $b$ are some non-zero real numbers. It is given that $f(x)$ has exactly three negative real roots.
(A) Determine whether each of the numbers $a, b$ is positive or negative. You must give adequate justification for your answer.
(B) Now show that $f(x)$ has at least one positive root. You can use your answer from part (A), but you can also do this part independently.

## Q4 (9 points)

Consider the matrix

$$
A=\left[\begin{array}{ll}
-61 & 45 \\
-84 & 62
\end{array}\right]
$$

(a) Write down the characteristic equation of $A$.
(b) Solve the equation to find the eigenvalues of $A$.
(c) Now find the eigenvector corresponding to each eigenvalue. The components of your eigenvectors should be integers.
(d) Make a $2 \times 2$ matrix $P$ by using the eigenvectors as columns. Write down the diagonal matrix $D$ by listing your eigenvalues in the same order as the eigenvectors.
(e) Find $P^{-1}$ using the adjoint formula.
(f) Now find the matrices $P^{-1} A$ and $D P^{-1}$ by direct multiplication. They must come out to be equal because $P^{-1} A P=D$. If they do not, then you have made a mistake somewhere.

## Q5 (6 points)

Consider the $2 \times 2$ matrix

$$
A=\left[\begin{array}{rr}
-71 & 96 \\
x & 65
\end{array}\right]
$$

and the vector $v=\left[\begin{array}{l}3 \\ 2\end{array}\right]$.
It is given to you that $v$ is an eigenvector of $A$ for some eigenvalue $\lambda$. Find $x$ and $\lambda$.
(Hint: use the definition of an eigenpair.)

## Q6 (8 points)

Consider the polynomial

$$
f(x)=5 x^{3}+m x^{2}+21 x+5
$$

It is given to you that $2-i$ is a root of $f(x)$.
(a) Find the value of $m$.
(b) Now find all the remaining roots of $f(x)$.

Q7 (5 points)

Let $z$ be a complex number such that
(1) $|z|=1$, and
(2) $|z-i|=2$.

Find $z$ in Cartesian form.

Hint: let $z=x+i y$ where $x$ and $y$ are the real and imaginary parts of $z$. Now substitute this into (1) and (2) and try to find $x$ and $y$.

Note:
-> If there is more than one such $z$, then you must find them all.
-> Remember that there is a single number $z$ which satisfies both conditions (1) and (2). You are asked to find that $z$. This is one single question; not two.

