Solutions to the Midterm

November 21, 2020

Q1. Consider the 4×4 matrix

$$M = \begin{bmatrix} 4 & x & 1 & 3 \\ 2 & -1 & 3 & -2 \\ 0 & 1 & 5 & 4 \\ 6 & 2 & -8 & 1 \end{bmatrix}.$$

It is given to you that the cofactor $C_{4,3} = 58$. Find the value of x.

ANSWER:

To find the minor $M_{4,3}$, we remove row 4 and column 3 from M. This gives

$$M_{4,3} = \det \begin{bmatrix} 4 & x & 3 \\ 2 & -1 & -2 \\ 0 & 1 & 4 \end{bmatrix}.$$

Expanding by row 1, we get

$$\begin{aligned} M_{4,3} &= 4 \begin{vmatrix} -1 & -2 \\ 1 & 4 \end{vmatrix} - x \begin{vmatrix} 2 & -2 \\ 0 & 4 \end{vmatrix} + 3 \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \\ &= 4 \times (-2) - 8x + 3 \times 2 = -8x - 2. \end{aligned}$$

Since $4+3=7$ is odd,

$$C_{4,3} = -M_{4,3} = 8\,x + 2 = 58.$$

Hence

$$8\,x = 56 \implies x = 7.$$

Q2. Consider the system of equations

$$3x + 2y = 4,$$
$$5x + 4y = 10.$$

In matrix form, it can be written as

$$\underbrace{\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}}_{C} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ 10 \end{bmatrix}}_{b}.$$

Find x and y using Cramer's rule.

Answer: We have $\det C = 3 \times 4 - 2 \times 5 = 2$.

We get the matrix C_x (respectively C_y) by replacing the first (respectively second) column of C by b. Hence

$$\det C_x = \det \begin{bmatrix} 4 & 2 \\ 10 & 4 \end{bmatrix} = 4 \times 4 - 2 \times 10 = -4,$$

$$\det C_y = \det \begin{bmatrix} 3 & 4 \\ 5 & 10 \end{bmatrix} = 3 \times 10 - 4 \times 5 = 10.$$

Hence

$$x = \frac{\det C_x}{\det C} = \frac{-4}{2} = -2, \quad y = \frac{\det C_y}{\det C} = \frac{10}{2} = 5.$$

Q3. Consider the same system of equations as in Q2.

(a) Find C^{-1} using the adjoint formula.

(b) Now find the values of x and y using the formula

$$\begin{bmatrix} x \\ y \end{bmatrix} = C^{-1}\, b.$$

ANSWER:

Recall that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Hence

$$C^{-1} = \frac{1}{2} \begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix}.$$

Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = C^{-1}\, b = \begin{bmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 10 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$$

Hence the solution is

$$x = -2, \quad y = 5.$$

Q4. Let

$$M = \begin{bmatrix} 1 & -2 & -2 & 3 \\ -3 & 6 & 1 & -14 \\ -1 & 2 & 1 & -4 \end{bmatrix}$$

Convert M to RREF. You should clearly write down your row-operations and intermediate matrices.

ANSWER: The sequence of row-operations is as follows:

$$M = \begin{bmatrix} 1 & -2 & -2 & 3 \\ -3 & 6 & 1 & -14 \\ -1 & 2 & 1 & -4 \end{bmatrix} \xrightarrow{R_2+3R_1, R_3+R_1}$$

$$\begin{bmatrix} 1 & -2 & -2 & 3 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -1 & -1 \end{bmatrix} \xrightarrow{R_2/(-5), -R_3}$$

$$\begin{bmatrix} 1 & -2 & -2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & -2 & -2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+2R_2}$$

$$\begin{bmatrix} 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Leftarrow \text{RREF}$$

Q5. Write the complex number

$$z = (\imath^{23} + 2\,\imath^{-34})\,\overline{(3 + 5\,i)} + \frac{1}{i}$$

in Cartesian form. The ‘bar’ indicates complex conjugate.

ANSWER: We have

$$23 = 4 \times 5 + 3 \implies \imath^{23} = \imath^{3} = -i,$$
$$-34 = 4 \times (-9) + 2 \implies \imath^{-34} = \imath^{2} = -1.$$

Also

$$\frac{1}{i} = \frac{1.(-i)}{i.(-i)} = \frac{-i}{-i^2} = \frac{-i}{1} = -i.$$

Hence

$$\begin{aligned} z &= (-2 - i)\,(3 - 5\,i) - i \\ &= -6 + 10\,i - 3\,i + 5\,i^2 - i \\ &= -6 - 5 + 6\,i = -11 + 6\,i. \end{aligned}$$

Q6. Let

$$z_1 = -3 + 5\,i, \quad \text{and} \quad z_2 = 1 - 8\,i.$$

(a) Express the complex number

$$\frac{z_1}{z_2}$$

in Cartesian form.

(b) Find

$$|z_1|, \quad |z_2|, \quad \left|\frac{z_1}{z_2}\right|$$

and verify that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

ANSWER: We have

$$\frac{z_1}{z_2} = \frac{-3+5i}{1-8i} = \frac{(-3+5i)(1+8i)}{(1-8i)(1+8i)}$$

$$= \frac{-3-24i+5i+40i^2}{1^2+8^2}$$

$$= \frac{-3-40-19i}{65} = -\frac{43}{65} - i\,\frac{19}{65}.$$

Now,

$$|z_1| = \sqrt{3^2 + 5^2} = \sqrt{9 + 25} = \sqrt{34},$$
$$|z_2| = \sqrt{1^2 + 8^2} = \sqrt{1 + 64} = \sqrt{65}.$$

And

$$\begin{aligned}
\left|\frac{z_1}{z_2}\right| &= \sqrt{\left(\frac{43}{65}\right)^2 + \left(\frac{19}{65}\right)^2} \\
&= \frac{\sqrt{43^2 + 19^2}}{65} = \frac{\sqrt{1849 + 361}}{65} = \frac{\sqrt{2210}}{65} \\
&= \frac{\sqrt{34 \times 65}}{65} = \frac{\sqrt{34}\sqrt{65}}{(\sqrt{65})^2} = \frac{\sqrt{34}}{\sqrt{65}} = \frac{|z_1|}{|z_2|}.
\end{aligned}$$

Q7. It is given to you that the matrix

$$M = \begin{bmatrix} 1 & p+2q+1 & 0 \\ 0 & 2-q & 1 \\ 0 & 0 & p+5q \end{bmatrix}$$

is in RREF. Find the values of p and q. Then write down the matrix obtained by substituting those values.

You should clearly express your reasoning process. Use English sentences if necessary. Also, if there is more than one solution, then you must find them all.

Answer: We have

$$M = \begin{bmatrix} 1 & p+2q+1 & 0 \\ 0 & 2-q & 1 \\ 0 & 0 & p+5q \end{bmatrix}$$

Notice that the $(3,3)$ entry must be zero. The reasoning is as follows: the entry must be 0 or 1. But if it were 1, it would be a leading 1. But this is impossible, because there is a non-zero entry above it, which is not allowed in an RREF.

Hence,

$$p+5q=0.$$

Now, the $(2,2)$ entry can only be 1 or 0. Let us check both possibilities:

Suppose $2-q=1$. But then

$$2-q=1,\ p+5q=0 \implies q=1, p=-5.$$

If we substitute this solution into M, we get

$$M=\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which is not in RREF, because the $(1,2)$ entry is not zero.
Hence, $2-q=1$ is impossible.

The only possibility is $2 - q = 0$. Then

$$2 - q = 0,\ p + 5q = 0 \implies q = 2, p = -10.$$

If we substitute this into M, we get

$$M = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which is in fact in RREF. Hence $p = -10, q = 2$ is the only possible answer.