MATH 1210 A03 Final Exam Fall 2021

Date and Time: January 20 at 12:30 PM   Duration: 120+30 minutes

Attempt all questions and show your work. Simplify your answers as much as possible.

1. (8 points) Let i be the complex number for which $i^2 = -1$. Use mathematical induction to prove that

$$1 + (1+i) + (1+i)^2 + (1+i)^3 + \cdots + (1+i)^n = i\big[1 - (1+i)^{n+1}\big]$$

for all positive integers $n \geqslant 1$,

2. Consider the polynomial equation of $P(x) = 0$ where

$$P(x) = x^4 - x^3 - x^2 - x - 2\,.$$

   (a) (3 points) What are the number of possible positive and negative real zeros of $P(x)$?

   (b) (7 points) Find all zeros of $P(x)$.

3. Consider the plane $\Pi\,:\,2x - y + z = 1$ and the line $\ell\,:\,\dfrac{x-2}{-1} = \dfrac{y-3}{-2} = \dfrac{z+1}{1}$.

   (a) (4 points) Find the intersection point of the plane Π and the line ℓ.

   (b) (4 points) Find an equation for the plane Π_1 through the origin and parallel to the plane Π.

4. (10 points) Find all **basic** solutions of the homogeneous linear system

$$\begin{array}{cccccccc} -x_1 & +2x_2 & +x_3 & -2x_4 & & = & 0 \\ & & +x_3 & -2x_4 & & = & 0 \\ -2x_1 & +4x_2 & & & +x_5 & = & 0 \end{array}.$$

5. (7 points) Let a be a negative real number. Use Cramer's rule to find *only* the value of z.

$$\begin{array}{l} ax + z = a - 2 \\ x + y + z = -3 \\ 3x - y + 2z = 1 \end{array}.$$

6. Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & \frac{11}{3} \\ 0 & 3 & 2 \end{pmatrix}$.

   (a) (7 points) Find inverse of A^T.

   (b) (3 points) Use part (a) to solve the linear system $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$.

7. (7 points) Let $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}$. First show that A is invertible and then find the entry of A^{-1} that lies in the fourth row and first column. Show your work. (You do not need to find all entries of A^{-1})

8. Let $\mathbf{u} = \langle -1, 0, 1\rangle$, $\mathbf{v} = \langle 3, 2, 1\rangle$ and $\mathbf{w} = \langle 0, 1, 2\rangle$.

   (a) (6 points) Prove that the vectors $\mathbf{u} - \mathbf{v}$, $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ are linearly independent. .

   (b) (8 points) Prove that the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent and express $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

9. (8 points) A linear transformation T_1 from $\mathbb{R}^2$ to $\mathbb{R}^2$ maps vectors $\mathbf{v} = \langle v_1, v_2 \rangle$ to $\mathbf{v}' = \langle v_1', v_2' \rangle$ according to T_1 : $\begin{array}{l} v_1' = 5v_1 + 3v_2 \\ v_2' = 2v_1 + v_2 \end{array}$.
   Also a linear transformation T_2 from $\mathbb{R}^2$ to $\mathbb{R}^2$ is defined by $T_2(\langle v_1, v_2 \rangle) = \langle v_1 + 2v_2 \ , \ 6v_1 - v_2 \rangle$.
   Find associated matrices for linear transformations $2T_1 - 3T_2$ and T_1^{-1}.

10. Let $A = \begin{pmatrix} 0 & 2 \\ 2 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 0 \\ 4 & -1 & 0 \\ 7 & 3 & 5 \end{pmatrix}$.
    (a) (6 points) Find all eigenvalues of A and all eigenvalues of B.
    (b) (7 points) Find two eigenvectors of A such that they are perpendicular. Show your work.