
Attempt all questions and show your work. Simplify your answers as much as possible. You may use your textbook, and the files available on UM learn. You may use the formulas at the end of this exam.

[7] 1. Let x be a real number, and let A be the matrix

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Use proof by induction to show that, for all positive integers n, we have

$$A^{2n} = \begin{pmatrix} 1 & -2n & nx \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

[7] 2. Find all the solutions to the equation $z^2 = \dfrac{i}{1-i}$. Express your answer in Exponential form.

3. Let $P(x)$ be the polynomial $3x^4 + 13x^3 + 11x^2 - 13x - 14$.

[4] (a) List all the possible rational solutions of $P(x) = 0$.

[3] (b) Use The Bounds theorem to reduce the list from part (a)

[3] (c) Use the fact that $x = 1$ is a zero of $P(x)$ and Descartes' Rule to reduce the list from part (b).

[3] (d) Find the remainder when $P(x)$ is divided by $(x - i)$.

[6] 4. Consider the line

$$L : x = 3t, \quad y = 1 + t, \quad z = 4 + 5t, \quad t \in \mathbb{R}$$

and the plane

$$\Pi : -3x + y - 2z = 9.$$

Find the symmetric equation of a line M parallel to the plane Π and perpendicular to the line L and passing through the point $(1, 2, 3)$.

[7] 5. Find all solutions to the linear system

$$\begin{aligned} 3x - 2y + z - 4w &= 0, \\ 6x - 3y + 2z - 5w &= 0, \\ -3x - y + 6z - 5w &= 0. \end{aligned}$$

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[4] 6. Let A, B and C be the matrices

$$A = \begin{pmatrix} 1 & 5 & -1 \\ 0 & -7 & 8 \\ 0 & 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} -\sqrt{2} & 0 & 0 \\ -1 & \sqrt{2} & 0 \\ -24 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} \sqrt{10} & 0 & 0 \\ 0 & 1 & \sqrt{10} \\ 3 & 0 & \sqrt{10} \end{pmatrix}$$

Calculate $|ABC|$.

[9] 7. Let x be a real number, and let A be the matrix

$$A = \begin{pmatrix} 2 & -1 & -2 \\ 4 & -1 & 1 \\ 8 & -2 & 2+2x \end{pmatrix}.$$

Determine for which values of x the matrix A is invertible, and find the inverse in that case.

8. Let u be the vector

$$u = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

[4] (a) Show that the transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$$T(v) = (u \times v) + v$$

is linear. (You may use the convention $(u \times v) + v = u \times v + v$.)

[5] (b) Show that the matrix for T is of the form

$$A = \begin{pmatrix} x & 0 & y \\ 0 & x & -y \\ -y & y & x \end{pmatrix},$$

and determine x and y.

[5] (c) Determine the eigenvalues of T. (This should result in 1 real eigenvalue.)
HINT: When you try to solve the equation $|T - \lambda I_3| = 0$ it might help to substitute $z = (1 - \lambda)$.
If you were unable to solve (b), you may use $x = \sqrt{2}$ and $y = 1$. WARNING: These are not the correct values, and for these numbers, the hint above does not work.

[3] (d) Find one real eigenpair of T.

9. For each of the following statements, determine if it's true or false. Briefly justify your answer or give a counter-example. Any answer without justification will be given a mark of 0.

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[2] (a) Let A be a square matrix, and let (λ, v) be an eigenpair for A. If t is a non-zero real number, then (λ, tv) is an eigenpair for A.

[2] (b) Let A be a square matrix, and let (λ, v) be an eigenpair for A. If t is a non-zero real number, then $(t\lambda, v)$ is an eigenpair for A.

[2] (c) If A is an upper triangular matrix, and A^T is upper triangular, then A is diagonal.

[2] (d) If A is an upper triangular matrix, and A^T is lower triangular, then A is diagonal.

[2] (e) Let A be any square matrix, and set $B = A + A^T$. Then, all eigenvalues of B are real. (HINT: What is B^T?)

Formulas & Conventions:

quadratic formula: $x_\pm = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

sum of integers: $\sum_{n=1}^{m} n = \frac{m(m+1)}{2}$.

sum of squares: $\sum_{n=1}^{m} n^2 = \frac{m(m+1)(2m+1)}{6}$.

sum of cubes: $\sum_{n=1}^{m} n^3 = \frac{m^2(m+1)^2}{4}$.

Euler's formula: $e^{it} = \cos(t) + i\sin(t)$.

Eigenpairs: If λ is an eigenvalue for a matrix A, and v is the corresponding eigenvector, then we write (λ, v) for the eigenpair. That is, we write the number first and the vector second.

Principal argument: The principal argument of a complex number always lies in $(-\pi, \pi]$.