
This is a **practice** exam, to help prepare you for the final. Attempt all questions and show your work. Simplify your answers as much as possible. You may use your textbook, and the files available on UM learn. You may use the formulas at the end of this exam.

[6] 1. Consider the polynomial

$$P(x) = x^6 - 2\sqrt{2}x^3 + 4.$$

Find all roots of $P(x) = 0$. Give your answers in exponential form. Use the principal value for the argument.

Solution: First, we substitute $y = x^3$, and solve the equation

$$y^2 - 2\sqrt{2}x + 4 = 0.$$

The quadratic formula gives

$$y_\pm = \frac{2\sqrt{2} \pm \sqrt{8-16}}{2} = \sqrt{2} \pm \sqrt{2}i = \sqrt{2}(1 \pm i).$$

We want to convert $y_\pm$ to exponential form, so we first calculate the modulus

$$|y_\pm|^2 = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2.$$

The argument is of $y_\pm$ is $\pm\pi/4$, thus we obtain.

$$\sqrt{2}(1 \pm i) = 2e^{\pm i\pi/4}.$$

We now recall that we set $y = x^3$, we must thus find all solutions to

$$x^3 = y_+ = 2e^{i\pi/4}, \qquad \text{and ,} \qquad x^3 = y_- = 2e^{-i\pi/4}.$$

This leads to

$$2^{1/3}e^{i\frac{\pi}{12}}, \quad 2^{1/3}e^{i\frac{3\pi}{4}}, \quad 2^{1/3}e^{-i\frac{7\pi}{12}},$$
$$2^{1/3}e^{-i\frac{\pi}{12}}, \quad 2^{1/3}e^{-i\frac{3\pi}{4}}, \quad 2^{1/3}e^{i\frac{7\pi}{12}}.$$

[7] 2. Let x be a real number, and let A be the matrix

$$\begin{pmatrix} 2 & x \\ 0 & 1 \end{pmatrix}.$$

Use proof by induction to show that, for all positive integers n, we have

$$A^n = \begin{pmatrix} 2^n & (2^n - 1)x \\ 0 & 1 \end{pmatrix}.$$

Solution: For all positive integers n, let P_n be the statement

$$A^n = \begin{pmatrix} 2^n & (2^n-1)x \\ 0 & 1 \end{pmatrix}.$$

Base case: $(n = 1)$. We calculate

$$\begin{pmatrix} 2^1 & (2^1-1)x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & x \\ 0 & 1 \end{pmatrix} = A.$$

This is the statement P_1, thus the base case is established.
Inductive step: $(P_n \Rightarrow P_{n+1})$. Let k be an arbitrary positive integer, and assume that P_k holds, i.e.

$$A^k = \begin{pmatrix} 2^k & (2^k-1)x \\ 0 & 1 \end{pmatrix}.$$

We must show that P_{k+1} then follows, i.e.

$$A^{k+1} \stackrel{?}{=} \begin{pmatrix} 2^{k+1} & (2^{k+1}-1)x \\ 0 & 1 \end{pmatrix}.$$

We thus calculate

$$\begin{aligned} A^{k+1} = A^k A &= \begin{pmatrix} 2^k & (2^k-1)x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & x \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & 2^k x + (2^k-1)x \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & (2^k + 2^k - 1)x \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & (2^{k+1}-1)x \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

This is the desired equality. Thus, by the principle of mathematical induction, we have that the statement P_n is true for all positive integers n.

[3] 3. Let A and B be the matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 7 \\ 0 & 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} -3 & 0 & 0 \\ 11 & 1 & 0 \\ -12 & 5 & 2 \end{pmatrix}$$

Calculate $|AB|$.

Solution: Because the matrix A is upper triangular, and the matrix B is lower triangular, we can use Theorem 7.2 from the textbook to calculate their determinants: $|A| = 1\cdot(-4)\cdot 2 = -8$ and $|B| = (-3)\cdot 1\cdot 2 = -6$. We then use Theorem 7.6 and get $|AB| = |A||B| = (-8)\cdot(-3) = 24$.

It is also possible to first calculate AB and then calculate $|AB|$ directly, but this would be much more work.

[5] 4. Evaluate the sum

$$\sum_{j=3}^{k} 2j(3j-14).$$

(This problem has nothing to do with induction.)

Solution: We first shift the index and then expand the brackets

$$\sum_{j=3}^{k} 2j(3j-14) = \sum_{j=1}^{k-2} 2(j+2)(3j-8) = \sum_{j=1}^{k-2} \left(6j^2 - 4j - 32\right).$$

We then split the sum into three and apply the formulas below:

$$\begin{aligned}\sum_{j=1}^{k-2} \left(6j^2 - 4j - 32\right) &= 6\sum_{j=1}^{k-2} j^2 - 4\sum_{j=1}^{k-2} j - 32\sum_{j=1}^{k-2} 1\\ &= (k-2)(k-1)(2k-3) - 2(k-2)(k-1) - 32(k-2)\\ &= (k-2)(k-1)(2k-5) - 32(k-2)\\ &= 2k^3 - 11k^2 - 13k + 54.\end{aligned}$$

[4] 5. Let t be a real number. Calculate the determinant of

$$A = \begin{pmatrix} 1 & e^{it} & e^{-it} \\ -1 & e^{it} & e^{-it} \\ 0 & e^{-it} & e^{it} \end{pmatrix}$$

by expanding along the first column. Give your answer in Cartesian form.

Solution: We obtain

$$\begin{aligned}|A| &= \det\begin{pmatrix} e^{it} & e^{-it} \\ e^{-it} & e^{it}\end{pmatrix} - (-1)\det\begin{pmatrix} e^{it} & e^{-it} \\ e^{-it} & e^{it}\end{pmatrix} \\ &= 2\det\begin{pmatrix} e^{it} & e^{-it} \\ e^{-it} & e^{it}\end{pmatrix} \\ &= 2(e^{2it} - e^{-2it}) \\ &= 2(\cos(2t) + i\sin(2t) - \cos(-2t) - i\sin(-2t)) \\ &= 2(\cos(2t) + i\sin(2t) - \cos(2t) + i\sin(2t)) \\ &= i4\sin(2t).\end{aligned}$$

This is in Cartesian form: $a + ib$, with $a = 0$ and $b = 4\sin(2t)$.

[5] 6. For each of the following matrices, find the inverse, if possible.

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 8 & 1 \\ 0 & -4 & 7 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & -11 \\ 2 & 4 & 6 \\ 3 & 6 & -3 \end{pmatrix}, \qquad C = \begin{pmatrix} 6 & -2 \\ 2 & \frac{1}{2} \end{pmatrix}.$$

Solution: The matrices A and B are not invertible, because $\det(A) = 0$ and $\det(B) = 0$. The inverse of C can be computed as follows

$$\left(\begin{array}{cc|cc} 6 & -2 & 1 & 0 \\ 2 & \frac{1}{2} & 0 & 1 \end{array}\right)$$

We multiply row 2 by 2.

$$\left(\begin{array}{cc|cc} 6 & -2 & 1 & 0 \\ 4 & 1 & 0 & 2 \end{array}\right)$$

Add 2 times row 2 to row 1.

$$\left(\begin{array}{cc|cc} 14 & 0 & 1 & 4 \\ 4 & 1 & 0 & 2 \end{array}\right)$$

Divide row 1 by 14.

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{14} & \frac{2}{7} \\ 4 & 1 & 0 & 2 \end{array}\right)$$

Subtract 4 times row 1 from row 2.

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{14} & \frac{2}{7} \\ 0 & 1 & -\frac{2}{7} & \frac{6}{7} \end{array}\right)$$

The inverse of C is thus

$$C^{-1} = \frac{1}{7}\begin{pmatrix} \frac{1}{2} & 2 \\ -2 & 6 \end{pmatrix}$$

It is also possible to calculate the inverse of C by the adjoint method.

[6] 7. Let v and w be the vectors

$$v = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}, \qquad w = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}.$$

Find all vectors

$$u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

such that the equation $v \times u = w$ holds.

Solution: We calculate

$$v \times u = \begin{pmatrix} -6y - z \\ 6x - 2z \\ x + 2y \end{pmatrix}$$

We thus obtain the linear system

$$\begin{aligned} -6y - z &= 2, \\ 6x - 2z &= -2, \\ x + 2y &= -1. \end{aligned}$$

The augmented matrix becomes

$$\left(\begin{array}{ccc|c} 0 & -6 & -1 & 2 \\ 6 & 0 & -2 & -2 \\ 1 & 2 & 0 & -1 \end{array}\right)$$

Gaussian elimination leads to the solutions

$$u = \begin{pmatrix} x \\ -\frac{x+1}{2} \\ 3x + 1 \end{pmatrix}.$$

FORMULAS:

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quadratic formula: $x_\pm = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

sum of integers: $\sum_{n=1}^m n = \frac{m(m+1)}{2}$.

sum of squares: $\sum_{n=1}^m n^2 = \frac{m(m+1)(2m+1)}{6}$.

sum of cubes: $\sum_{n=1}^m n^3 = \frac{m^2(m+1)^2}{4}$.

Euler's formula: $e^{it} = \cos(t) + i\sin(t)$.

This practice exam does not cover all the topics that will be covered on the final.