MATH 1210 Midterm Exam Fall 2021

Date and Time: October 25 at 6:00 PM   Duration: 60+15 minutes

Attempt all questions and show your work. Simplify your answers as much as possible.

1. (8 points) Consider the matrix
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
Use induction on n to show that for all integers $n \geqslant 1$ we have
$$A^{4n} = \mathrm{I}_2,$$
where I_2 is the 2×2 identity matrix.

2. (5 points) Write the following sum in sigma notation. Start summation index from 1.
$$-3a^4 + 8a^5 - 13a^6 + 18a^7 - 23a^8 + \cdots + 48a^{13}$$

3. (11 points) Simplify the complex expression below and then write it in Cartesian form.
$$\left[\frac{\overline{1+3i}}{-2+i}\right]^{100}$$

4. (10 points) Consider the polynomial
$$P(x) = x^3 + 3x^2 + 4x + 12.$$
Given is that $2i$ is a root of $P(x) = 0$. Find the remaining roots of $P(x) = 0$, and write $P(x)$ as a product of linear factors.

5. True or false? Briefly justify your answer.
   (a) (2 points) There exist two matrices of size 3×3, say A and B, such that $AB = BA$.
   (b) (2 points) There exist two square matrices of the same size, A and B, such that $AB \neq BA$.
   (c) (2 points) If A is a matrix of any size, then the expression $A^T A$ exists.
   (d) (2 points) Any polynomial of degree n can be written as a product of exactly n linear factors with real coefficients.

6. (8 points) Consider three vectors
$$\mathbf{u} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + a\hat{\mathbf{k}}, \qquad \mathbf{v} = 2\hat{\mathbf{i}} + b\hat{\mathbf{j}} + \hat{\mathbf{k}}, \qquad \mathbf{w} = -4\hat{\mathbf{i}} + (b-2)\hat{\mathbf{j}} + (b+8)\hat{\mathbf{k}},$$
where a and b are integers. Find values of a and b such that $\mathbf{u}$ is perpendicular to $\mathbf{v}$ and $2\mathbf{u} + \mathbf{w} = \mathbf{v}$.

7. Consider the lines
$$\ell_1 : \quad x = 4 - 3t,\ y = 2t,\ z = 1 + 5t\ , \qquad \ell_2 : \quad \frac{x-4}{4} = \frac{y}{1} = \frac{z-1}{2}\ .$$
   (a) (3 points) Find the intersection point of the lines ℓ_1 and ℓ_2. Justify your answer.
   (b) (7 points) Find an equation of the plane that contains both ℓ_1 and ℓ_2.