MATH 1210 Assignment 3 Fall 2023 **Due date: November 27, 5:00 PM**

Attempt all questions and show all your work. Some or all questions will be marked.

1. Consider the lines

$$\ell_1: \quad x = 3 + t, \quad y = 1 - t, \quad z = 3t, \quad t \in R,$$

$$\ell_2: \quad \frac{x}{2} = y + 2 = \frac{z-5}{-1},$$

the plane

$$\Pi: \quad x + y + 1 = 0,$$

and the point $P(2, 1, 0)$.

(a) Find the point Q of intersection of the lines ℓ_1 and ℓ_2.

(b) Find an equation of the plane containing both lines ℓ_1 and ℓ_2.

(c) Find an equation of the plane which is perpendicular to the plane Π and passes through the points P and Q.

(d) Find both parametric and symmetric equations (if possible) for the line which is parallel to the line ℓ_1 and passes through the origin.

Solution:

(a) Suppose that the point Q has coordinates (a, b, c). Since Q is on line ℓ_1, then a, b and c have to satisfy

$$a = 3 + t_1, \quad b = 1 - t_1, \quad c = 3t_1, \quad \text{for some } t_1 \in R.$$

Similarly, since Q is on line ℓ_2, then a, b and c have to satisfy

$$\frac{a}{2} = b + 2 = \frac{c-5}{-1} =: t_2, \quad \text{for some } t_2 \in R.$$

Hence,

$$3 + t_1 = 2t_2,\ 1 - t_1 = -2 + t_2,\ 3t_1 = 5 - t_2 \quad \iff \quad t_1 - 2t_2 = -3,\ t_1 + t_2 = 3,\ 3t_1 + t_2 = 5.$$

This implies that $t_1 = 1$ and $t_2 = 2$, and so $Q(4, 0, 3)$.

(b) First, note that vectors $\mathbf{u}_1 = \langle 1, -1, 3\rangle$ and $\mathbf{u}_2 = \langle 2, 1, -1\rangle$ are parallel to ℓ_1 and ℓ_2, respectively. If a plane contains both lines ℓ_1 and ℓ_2, then it has to be parallel to the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, and so its normal vector $\mathbf{n}$ has to be perpendicular to both them. Therefore, $\mathbf{n}$ can be taken as the cross product of $\mathbf{u}_1$ and $\mathbf{u}_2$:

$$\mathbf{n} = \mathbf{u}_1 \times \mathbf{u}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 3 \\ 2 & 1 & -1 \end{vmatrix} = \langle -2, 7, 3\rangle$$

Hence, this plane has the following equation (using the fact that it passes through Q, but any other point on ℓ_1 or ℓ_2 can be used):

$$-2(x-4) + 7y + 3(z-3) = 0 \quad \iff \quad -2x + 7y + 3z - 1 = 0$$

(c) Note that $\langle 1,1,0\rangle$ is a normal vector to the plane Π. Let $\mathbf{n}$ denote a normal vector to the plane that we are looking for in this part. Since this plane is perpendicular to Π, normal vectors to these planes are perpendicular, i.e., $\mathbf{n} \perp \langle 1,1,0\rangle$. Also, since this plane passes through P and Q, the vector $\overrightarrow{PQ}$ is parallel to it, and so it is perpendicular to $\mathbf{n}$: $\mathbf{n} \perp \overrightarrow{PQ}$. Hence, we can take $\mathbf{n}$ to be the cross product of $\langle 1,1,0\rangle$ and $\overrightarrow{PQ} = \langle 2,-1,3\rangle$:

$$\mathbf{n} = \langle 1,1,0\rangle \times \langle 2,-1,3\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 0 \\ 2 & -1 & 3 \end{vmatrix} = \langle 3,-3,-3\rangle$$

In fact, since $\mathbf{n} = 3\langle 1,-1,-1\rangle$, the vector $\langle 1,-1,-1\rangle$ is also a normal to the plane that we are looking for. Hence, using the fact that the plane passes through the point P, we get the following equation:

$$(x-2)-(y-1)-z=0 \quad \iff \quad x-y-z-1=0$$

(d) As we already discussed in part (b), the vector $\mathbf{u}_1 = \langle 1,-1,3\rangle$ is parallel to the line ℓ_1, and so it is also parallel to the line that we are looking for. Using the fact that this line passes through the origin we get the following parametric equations:

$$x=t, \quad y=-t, \quad z=3t, \quad t \in R.$$

Hence, the symmetric equations for this line are:

$$x = \frac{y}{-1} = \frac{z}{3}.$$

2. Consider the following linear system of equations.

$$\begin{cases} x-y+z-2w=1 \\ -x+y+z+w=-1 \\ -x+2y+3z-w=2 \\ x-4y-13z+8w=-8 \end{cases}$$

(a) Find the reduced row-echelon form (RREF) of the augmented matrix.

(b) Find all solutions of this system (i.e., determine the solution set).

Solution: (a) We reduce the augmented matrix of this system to RREF as follows

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & -2 & 1 \\ -1 & 1 & 1 & 1 & -1 \\ -1 & 2 & 3 & -1 & 2 \\ 1 & -4 & -13 & 8 & -8 \end{array}\right] \underset{\substack{R_2\to R_2+R_1 \\ R_3\to R_3+R_1 \\ R_4\to R_4-R_1}}{\longrightarrow} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & -2 & 1 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 4 & -3 & 3 \\ 0 & -3 & -14 & 10 & -9 \end{array}\right] \underset{R_4\to R_4+3R_3}{\longrightarrow}$$

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & -2 & 1 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 4 & -3 & 3 \\ 0 & 0 & -2 & 1 & 0 \end{array}\right] \underset{\substack{R_2\leftrightarrow R_3 \\ R_4\to R_4+R_2}}{\longrightarrow} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & -2 & 1 \\ 0 & 1 & 4 & -3 & 3 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \underset{\substack{R_1\to R_1-R_3/2 \\ R_2\to R_2-2R_3 \\ R_3\to R_3/2}}{\longrightarrow}$$

$$\left[\begin{array}{rrrr|r} 1 & -1 & 0 & -3/2 & 1 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \underset{R_1\to R_1+R_2}{\longrightarrow} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -5/2 & 4 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b) The general solution of this linear system is

$$x = \frac{5}{2}t + 4, \quad y = t + 3, \quad z = \frac{1}{2}t, \quad w = t, \quad t \in R.$$

Alternatively, this solution can be written as

$$x = \frac{5}{2}w + 4, \quad y = w + 3, \quad z = \frac{1}{2}w, \quad w \in R.$$

3. Let

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 1 \\ -1 & 2 & 3 & 0 & 4 \\ 1 & -4 & 0 & 8 & 5 \end{bmatrix}.$$

(a) While reducing A to RREF, a student in MATH 1210 started the reduction process by doing the following:

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 1 \\ -1 & 2 & 3 & 0 & 4 \\ 1 & -4 & 0 & 8 & 5 \end{bmatrix} \underset{\substack{R_2 \to R_2 + R_4 \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 + R_3}}{\rightarrow} \begin{bmatrix} 1 & -1 & 1 & 0 & 1 \\ 0 & -4 & 1 & 9 & 6 \\ 0 & 2 & 2 & -1 & 3 \\ 0 & -2 & 3 & 8 & 9 \end{bmatrix}.$$

Explain why this student will not get a correct answer and what mistake they made.

(b) Find the correct reduced row-echelon form (RREF) of A.

Solution: (a) The student tried to combine several elementary row operations into one step of the reduction process. However, this is only allowed if this step can then be expanded into a sequence of steps each using only one elementary row operation, which would not be possible in this case. Note that, no matter what the original rows R_2, R_3 and R_4 are, the sum of rows 2 and 3 after this erroneous reduction step will always be equal to row 4. So, the student effectively "lost" one row during this reduction.

(b) This matrix A can be reduced to RREF as follows

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 1 \\ -1 & 2 & 3 & 0 & 4 \\ 1 & -4 & 0 & 8 & 5 \end{bmatrix} \underset{\substack{R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1 \\ R_4 \to R_4 - R_1}}{\to} \begin{bmatrix} 1 & -1 & 1 & 0 & 1 \\ 0 & -1 & 2 & 1 & 2 \\ 0 & 1 & 4 & 0 & 5 \\ 0 & -3 & -1 & 8 & 4 \end{bmatrix} \underset{\substack{R_2 \to -R_2 \\ R_3 \to R_3 + R_2 \\ R_4 \to R_4 - 3R_2}}{\to}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & -2 & -1 & -2 \\ 0 & 0 & 6 & 1 & 7 \\ 0 & 0 & -7 & 5 & -2 \end{bmatrix} \underset{R_3 \to R_3 + R_4}{\to} \begin{bmatrix} 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 & 5 \\ 0 & 0 & -7 & 5 & -2 \end{bmatrix} \underset{\substack{R_3 \to -R_3 \\ R_4 \to R_4 - 7R_3}}{\to}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & -2 & -1 & -2 \\ 0 & 0 & 1 & -6 & -5 \\ 0 & 0 & 0 & -37 & -37 \end{bmatrix} \underset{R_4 \to R_4/(-37)}{\to} \begin{bmatrix} 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & -2 & -1 & -2 \\ 0 & 0 & 1 & -6 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \underset{\substack{R_2 \to R_2 + R_4 \\ R_3 \to R_3 + 6R_4}}{\to}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \underset{\substack{R_1 \to R_1 - R_3 \\ R_2 \to R_2 + 2R_3}}{\to} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \underset{R_1 \to R_1 + R_2}{\to}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

4. Consider the system

$$\begin{cases} x + y + 2z &= a \\ 2x + by + 4z &= 1 \\ y + bz = 1 \end{cases}$$

In each case, determine all real numbers a and b which give the indicated number of solutions, if possible. If no such a and b exist, give the reason why not.

(a) no solutions

(b) exactly one solution

(c) infinitely many solutions

(d) exactly two solutions

Solution: First of all, a linear system can only have 0, 1 or infinitely many solutions, and so the case (d) is impossible.

There are two slightly different approaches to solving this problem, and we'll discuss both of them.

Approach 1: Since this is a system with 3 equations and 3 unknowns, we immediately know that it will have a unique solution if the determinant of the coefficient matrix is not equal to 0. This determinant is

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & b & 4 \\ 0 & 1 & b \end{vmatrix} = 1 \cdot \begin{vmatrix} b & 4 \\ 1 & b \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 2 \\ 1 & b \end{vmatrix} = (b^2 - 4) - 2(b - 2) = b(b - 2).$$

So, if $b \neq 0$ and $b \neq 2$, then the system has a unique solution for any $a \in R$. We now consider the cases when $b = 0$ and $b = 2$ separately (since the rank of the coefficient matrix in each of these cases is less than 3, i.e., there will be at least one zero row to the left of the vertical line in the augmented matrix, we know that the system will have either no or infinitely many solutions for these values of b).

Case $b = 0$: If $b = 0$, the augmented matrix for the system can be reduced toward REF as follows

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 2 & 0 & 4 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & a-1 \\ 2 & 0 & 4 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_1]{} \left[\begin{array}{ccc|c} 1 & 0 & 2 & a-1 \\ 0 & 0 & 0 & 3-2a \\ 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{} \left[\begin{array}{ccc|c} 1 & 0 & 2 & a-1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3-2a \end{array}\right]$$

Hence, if $3 - 2a = 0$, then the system will have infinitely many solutions. If $3 - 2a \neq 0$, then the system will have no solutions.

Case $b = 2$: If $b = 2$, the augmented matrix for the system can be reduced toward REF as follows

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 2 & 2 & 4 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_1]{} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 0 & 0 & 1-2a \\ 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1-2a \end{array}\right]$$

Hence, if $1 - 2a = 0$, then the system will have infinitely many solutions. If $1 - 2a \neq 0$, then the system will have no solutions.

Summarizing the above, we conclude that we have the following cases: (a) the system has no solutions if $b = 0$ and $a \neq 3/2$, or if $b = 2$ and $a \neq 1/2$; (b) the system has a unique solution if $b \neq 0$ and $b \neq 2$ (a can be any real number); (c) the system has infinitely many solutions if $b = 0$ and $a = 3/2$, or if $b = 2$ and $a = 1/2$.

Approach 2: We set up the augmented matrix for the system right away and then start reducing it to REF noting possible cases:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 2 & b & 4 & 1 \\ 0 & 1 & b & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_1]{} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & b-2 & 0 & 1-2a \\ 0 & 1 & b & 1 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & b & 1 \\ 0 & b-2 & 0 & 1-2a \end{array}\right] \xrightarrow[R_3 \to R_3 - (b-2)R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & b & 1 \\ 0 & 0 & -b(b-2) & 3-2a-b \end{array}\right]$$

We now see that the system will have a unique solution if $b(b-2) \neq 0$. It will have infinitely many solutions if $b(b-2) = 0$ and $3 - 2a - b = 0$. Finally, the system will have no solutions if $b(b-2) = 0$ and $3 - 2a - b \neq 0$. One now arrives at the same summary of all possible cases as above.

5. Find all basic solutions to the following homogeneous system and express the general solution as a linear combination of these basic solutions:

$$\begin{cases} x_1 + 2x_2 - x_3 + x_4 + x_5 = 0 \\ -x_1 - 2x_2 + 2x_3 + x_5 = 0 \\ -x_1 - 2x_2 + 3x_3 + x_4 + 3x_5 = 0 \end{cases}$$

Solution: Since this is a homogeneous system, we can work with the coefficient matrix instead of the augmented matrix. The coefficient matrix can be reduced to RREF as follows:

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ -1 & -2 & 2 & 0 & 1 \\ -1 & -2 & 3 & 1 & 3 \end{bmatrix} \underset{\substack{R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1}}{\to} \begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 & 4 \end{bmatrix} \underset{R_3 \to R_3 - 2R_2}{\to} \begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_1 \to R_1 + R_2}{\to} \begin{bmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, variables x_2, x_4 and x_5 are parameters (x_1 and x_2 are leading variables), and the general solution can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_2 - 2x_4 - 3x_5 \\ x_2 \\ -x_4 - 2x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \quad x_2, x_4, x_5 \in R.$$

Basic solutions (written as column matrices) are:

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

6. Use Cramer's rule to solve for z without solving for any of the other variables:

$$\begin{cases} 5x + y - z = -7 \\ 2x - y - 2z = 6 \\ 3x + 2z = -8 \end{cases}$$

Solution: The coefficient matrix for this system is

$$A := \begin{pmatrix} 5 & 1 & -1 \\ 2 & -1 & -2 \\ 3 & 0 & 2 \end{pmatrix}.$$

Since the variable z corresponds to the third column, we replace the third column with the column consisting of constants on the right-hand side of all equations in the system to obtain

$$A_3 := \begin{pmatrix} 5 & 1 & -7 \\ 2 & -1 & 6 \\ 3 & 0 & -8 \end{pmatrix}.$$

Now,

$$\det(A) = 3 \cdot \begin{vmatrix} 1 & -1 \\ -1 & -2 \end{vmatrix} + 2 \cdot \begin{vmatrix} 5 & 1 \\ 2 & -1 \end{vmatrix} = 3(-2-1) + 2(-5-2) = -9 - 14 = -23,$$

where we used the cofactor expansion with respect to the third row. Also,

$$\det(A_3) = 3 \cdot \begin{vmatrix} 1 & -7 \\ -1 & 6 \end{vmatrix} - 8 \cdot \begin{vmatrix} 5 & 1 \\ 2 & -1 \end{vmatrix} = 3(6-7) - 8(-5-2) = -3 + 56 = 53.$$

Hence, using Cramer's Rule, we conclude that

$$z = \frac{\det(A_3)}{\det(A)} = -\frac{53}{23}.$$

7. Suppose that A and B are 2023×2023 matrices such that $AB = -BA$. Show that either $\det(A) = 0$ or $\det(B) = 0$.

Solution: Since $\det(\lambda A) = \lambda^n \det(A)$, for any $n \times n$ matrix A (where λ is a constant), we have

$$\det(AB) = \det(-BA) \iff \det(A)\det(B) = (-1)^{2023}\det(B)\det(A) \iff \det(A)\det(B) = 0.$$

Hence, either $\det(A) = 0$ or $\det(B) = 0$.

8. Show that $\det(A + B^T) = \det(A^T + B)$ for any $n \times n$ matrices A and B.

Solution: We have

$$\det(A + B^T) = \det\left[(A + B^T)^T\right] = \det(A^T + (B^T)^T) = \det(A^T + B),$$

and the proof is complete.

9. Determine whether each of the following sets of vectors is linearly dependent or linearly independent.
   (a) $\langle 1, -1\rangle$, $\langle 2, 1\rangle$, $\langle 1, 3\rangle$
   (b) $\langle 1, -1, 1\rangle$, $\langle 2, 1, 2\rangle$, $\langle 1, 3, -1\rangle$
   (c) $\langle 11, -1, 1, 0\rangle$, $\langle 0, 0, 0, 0\rangle$, $\langle 0, 3, -1, -2\rangle$

Solution:

(a) This is a set of 3 vectors with 2 components each. Since $3 > 2$, this set is necessarily linearly dependent.

(b) Since this is a set of 3 vectors with 3 components each, it is linearly dependent if and only if the determinant of the 3×3 matrix whose columns are the components of these vectors has value zero. Since

$$\begin{aligned}\begin{vmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \\ 1 & 2 & -1 \end{vmatrix} &= 1 \cdot \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} - 2 \cdot \begin{vmatrix} -1 & 3 \\ 1 & -1 \end{vmatrix} + 1 \cdot \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = (-1-6) - 2(1-3) + (-2-1) \\ &= -7 + 4 - 3 = -6 \neq 0,\end{aligned}$$

we conclude that the set is linearly independent.

(c) Any set of vectors containing the zero vector is linearly dependent.

10. Let $\mathbf{u} = \langle -1, 3, 0, 1\rangle$, $\mathbf{v} = \langle 3, 1, 2, 0\rangle$ and $\mathbf{w} = \langle 3, 11, 4, 3\rangle$. Either express the vector $\mathbf{y} = \langle 3, 2, 4, 0\rangle$ as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, or show that this is not possible.

Solution: If $\mathbf{y}$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, then there exist $c_1, c_2, c_3 \in R$ such that

$$\begin{aligned}\mathbf{y} = c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} &\iff \langle 3,2,4,0\rangle = c_1\langle -1,3,0,1\rangle + c_2\langle 3,1,2,0\rangle + c_3\langle 3,11,4,3\rangle \\ &\iff \langle 3,2,4,0\rangle = \langle -c_1+3c_2+3c_3, 3c_1+c_2+11c_3, 2c_2+4c_3, c_1+3c_3\rangle,\end{aligned}$$

and so we arrive at a linear system of equations with 3 unknowns and 4 equations. The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} -1 & 3 & 3 & 3 \\ 3 & 1 & 11 & 2 \\ 0 & 2 & 4 & 4 \\ 1 & 0 & 3 & 0 \end{array}\right].$$

Note that the columns of this matrix are $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{y}$ written as column vectors. We now start reducing this matrix to REF:

$$\left[\begin{array}{ccc|c} -1 & 3 & 3 & 3 \\ 3 & 1 & 11 & 2 \\ 0 & 2 & 4 & 4 \\ 1 & 0 & 3 & 0 \end{array}\right] \xrightarrow[R_4\to R_4+R_1]{R_1\to -R_1,\ R_2\to R_2+3R_1} \left[\begin{array}{ccc|c} 1 & -3 & -3 & -3 \\ 0 & 10 & 20 & 11 \\ 0 & 2 & 4 & 4 \\ 0 & 3 & 6 & 3 \end{array}\right] \xrightarrow[R_3\to R_3/2,\ R_4\to R_4/3]{} \left[\begin{array}{ccc|c} 1 & -3 & -3 & -3 \\ 0 & 10 & 20 & 11 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow[R_4\to R_4-R_3]{}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -3 & -3 \\ 0 & 10 & 20 & 11 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

Hence, the system is inconsistent, and so such numbers c_1, c_2 and c_3 do not exist. Therefore, $\mathbf{y}$ is not a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.