MATH 1210 Final Exam Fall 2023

Date and Time: December 14 at 1:30 PM   Duration: 2 Hours

Attempt all questions and show your work. Simplify your answers as much as possible.

1. (6 points) Evaluate the sum
$$\sum_{j=6}^{25} j(6-j)$$
using the identities
$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}, \qquad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$
You don't have to simplify expressions like $\dfrac{10 \cdot 11 \cdot 21}{6}$.

2. Consider the polynomial $P(x) = 2x^3 - 3x^2 + 3x - 1$.
   (a) (5 points) Find the remainder when $P(x)$ is divided by $2x + i$. Your answer should be written in Cartesian form.
   (b) (6 points) Find all zeros of $P(x)$

3. (6 points) Consider the line ℓ with parametric equations
$$x = -2, \; y = 1 - t, \; z = 3t + 4 \quad (t \in \mathbb{R}).$$
Find **all vectors** of length 2 that are parallel to the line ℓ.

4. Consider the following linear system of equations:
$$\begin{array}{rrrrrcl} x_1 & +5x_2 & & & -4x_5 & = & 0 \\ 3x_1 & +15x_2 & & +x_4 & -12x_5 & = & 0 \\ -x_1 & -5x_2 & -2x_3 & & +2x_5 & = & 0 \end{array}$$
   (a) (3 points) Without solving the system, determine the number of solutions for this system. Explain why your answer is correct without solving the system.
   (b) (9 points) Find all **basic** solutions of the system.

5. Consider 3×3 invertible matrices A and B such that $A = \begin{pmatrix} 4 & 1 & 1 \\ -1 & 7 & 3 \\ -4 & 1 & 2 \end{pmatrix}$ and $\det(B) = \dfrac{1}{2}$.
   (a) (6 points) Evaluate $\det(2B^T A^{-1})$.
   (b) (6 points) Use Cramer's rule to solve the linear system $AX = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ for x **only**, where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.
   (You are **not** asked to find values of y and z. No mark will be given for any other method.)

6. Let $\mathbf{u} = \langle 1, 7, -1 \rangle$, $\mathbf{v} = \langle -2, -7, 2 \rangle$ and $\mathbf{w} = \langle 1, 7, 0 \rangle$.
   (a) (4 points) Determine whether the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent or linearly independent.
   (b) (8 points) Express the vector $\mathbf{x} = \langle -5, 0, 7 \rangle$ as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Show your work.

7. Let
$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & -2 & -2 \\ 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) (4 points) Without finding A^{-1}, use the adjoint method to find the entry in second row and third column of A^{-1}.
(You are **not** asked to find all entries of A^{-1}. No mark will be given for any other method.)

(b) (7 points) Find $(A+B)^{-1}$.

8. Suppose that $\mathbf{u} = \langle 2, 1\rangle$, $\mathbf{v} = \langle -1, 1\rangle$, and let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a linear transformation such that $T(\mathbf{u}) = \langle 1, 0\rangle$ and $T(\mathbf{v}) = \langle 0, -1\rangle$.

   (a) (3 points) Find $T(2\mathbf{u} - \mathbf{v}) - \mathbf{u} + \mathbf{v}$.

   (b) (8 points) Find the matrix associated with T.

9. Let
$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (a) (5 points) Find all eigenvalues of A.

   (b) (8 points) Find all eigenvectors corresponding to the real eigenvalue of A.

10. (6 points) Let A be a 3×3 symmetric matrix with real entries such that all three eigenvalues of A are distinct. If it is known that $\mathbf{u} = \langle -\sqrt{5}, -2, 1\rangle$ and $\mathbf{v} = \langle 0, 1, 2\rangle$ are two eigenvectors corresponding to two different eigenvalues of A, find **all** eigenvectors corresponding to the third eigenvalue of A.