**UNIVERSITY OF MANITOBA**

DATE: May 21, 2009 — MIDTERM I
TITLE PAGE

COURSE: MATH 1210 — TIME: 50 minutes

EXAMINATION: Classical and Linear Algebra — EXAMINER: M. Davidson

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FAMILY NAME: (Print in ink) ____________________

GIVEN NAME(S): (Print in ink) ____________________

STUDENT NUMBER: ____________________

SIGNATURE: (in ink) ____________________
(I understand that cheating is a serious offense)

**INSTRUCTIONS TO STUDENTS:**

This is a 50 minute exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 5 pages of questions and also 1 blank page for rough work. Please check that you have all the pages. You may remove the blank page if you want, but be careful not to loosen the staple.

The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 50 points.

**Answer all questions on the exam paper** in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

| Question | Points | Score |
|---|---|---|
| 1 | 3 | |
| 2 | 5 | |
| 3 | 4 | |
| 4 | 6 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 12 | |
| Total: | 50 | |

[3] 1. Write the following in sigma notation:

$$3 + 7 + 11 + \ldots + (16n + 11)\,.$$

**Solution:**

$$\sum_{j=1}^{4n+3} (4j - 1)$$

Aside: To find the upper limit of summation, we want to find $j$ so that $4j - 1 = 16n + 11$ or $4j = 16n + 12$, so $j = 4n + 3$.

[5] 2. Evaluate $\displaystyle\sum_{j=1}^{12} (j-3)(j+2)$.

**Solution:**

$$\begin{aligned}
\sum_{j=1}^{12} (j-3)(j+2) &= \sum_{j=1}^{12} (j^2 - j - 6) \\
&= \sum_{j=1}^{12} j^2 - \sum_{j=1}^{12} j - 6\sum_{j=1}^{12} 1 \\
&= \left(\frac{(12)(13)(25)}{6}\right) - \left(\frac{(12)(13)}{2}\right) - 6(12) \\
&= 650 - 78 - 72 \\
&= 500
\end{aligned}$$

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[4] 3. For what value of $k$ does the polynomial $P(x) = 3x^4 + 5x^3 + kx^2 - x + 6$ have $x + 2$ as a factor?

**Solution:**

Since $(x + 2)$ is a factor of $P(x)$, then $-2$ is a root of $P(x)$. So

$$\begin{aligned} P(-2) &= 0 \\ 3(-2)^4 + 5(-2)^3 + k(-2)^2 - (-2) + 6 &= 0 \\ 3(16) + 5(-8) + 4k + 2 + 6 &= 0 \\ 48 - 40 + 4k + 8 &= 0 \\ 4k + 16 &= 0 \\ 4k &= -16 \\ k &= -4 \end{aligned}$$

[6] 4. Simplify $\dfrac{8 - 2i}{5 + 3i}$. Convert your answer into polar form.

**Solution:**

$$\begin{aligned} \frac{8-2i}{5+3i} &= \frac{(8-2i)(5-3i)}{(5+3i)(5-3i)} \\ &= \frac{40 - 24i - 10i + 6i^2}{25 - 15i + 15i - 9i^2} \\ &= \frac{34 - 34i}{34} \\ &= 1 - i \end{aligned}$$

Since $|1 - i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ and
$\arg(1 - i) = \frac{-\pi}{4}$, then
$1 - i = \sqrt{2}(\cos(\frac{-\pi}{4}) + i\sin(\frac{-\pi}{4}))$.

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[10] 5. Use induction to prove the following for all $n \geq 1$:

$$2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \cdots + 2 \cdot 3^{2n-1} = 3^{2n} - 1$$

**Solution:**

Let $P_n$ be the statement $2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \cdots + 2 \cdot 3^{2n-1} = 3^{2n} - 1$.

If $n = 1$ :
$2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \cdots + 2 \cdot 3^{2n-1} = 2 + 2 \cdot 3 = 8$
and $3^{2n} - 1 = 3^2 - 1 = 8$
So $P_1$ is true.

Assume that $P_k$ is true.
Then $2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \cdots + 2 \cdot 3^{2k-1} = 3^{2k} - 1$.

$\boxed{\text{Want: } 2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \cdots + 2 \cdot 3^{2(k+1)-1} = 3^{2(k+1)} - 1}$

Now,

$$\begin{aligned}
& 2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \cdots + 2 \cdot 3^{2(k+1)-1} \\
= & 2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \cdots + 2 \cdot 3^{2k+1} \\
= & 2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \cdots + 2 \cdot 3^{2k-1} + 2 \cdot 3^{2k} + 2 \cdot 3^{2k+1} \\
= & 3^{2k} - 1 + 2 \cdot 3^{2k} + 2 \cdot 3^{2k+1} \\
= & 3^{2k} + 2 \cdot 3^{2k} + 6 \cdot 3^{2k} - 1 \\
= & 9 \cdot 3^{2k} - 1 \\
= & 3^{2(k+1)} - 1
\end{aligned}$$

Hence $P_{k+1}$ is also true.

Since $P_1$ is true and $P_k$ implies $P_{k+1}$, by PMI, $P_n$ is true for all $n \geq 1$.

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[10] 6. Find all complex numbers $z$ such that $z^4 = -18\sqrt{2} - 18\sqrt{2}i$. Express your answer(s) in exponential form, using principle value for arguments.

**Solution:**

$|-18\sqrt{2} - 18\sqrt{2}i| = \sqrt{(-18\sqrt{2})^2 + (-18\sqrt{2})^2} = \sqrt{18^2 \cdot \sqrt{2}^2 + 18^2 \cdot \sqrt{2}^2} = \sqrt{18^2}\sqrt{4} = 18 \cdot 2 = 36$

$arg(-18\sqrt{2} - 18\sqrt{2}i) = \frac{-3\pi}{4}$

The exponential form is $-18\sqrt{2} - 18\sqrt{2}i = 36e^{i(\frac{-3\pi}{4})}$

Let $z = re^{i\theta}$, so $z^4 = r^4 e^{i(4\theta)}$.

So the equation $z^4 = -18\sqrt{2} - 18\sqrt{2}i$ becomes $r^4 e^{i(4\theta)} = 36e^{i(\frac{-3\pi}{4})}$ .

From this, we know that $r^4 = 36$ and $4\theta = \frac{-3\pi}{4} + 2n\pi$.

The modulus of all four roots is $r = \sqrt[4]{36} = \sqrt{6}$.

The arguments of the roots are

$$\begin{aligned} \theta &= (\frac{1}{4})(\frac{-3\pi}{4} + 2n\pi) \\ &= \frac{-3\pi}{16} + \frac{2n\pi}{4} \\ &= \frac{(-3 + 8n)\pi}{16} \end{aligned}$$

The arguments of the four roots can be found by finding angles for the values of $n = 0, 1, 2, 3$.

So $\theta_0 = \dfrac{-3\pi}{16}$,

$\theta_1 = \dfrac{5\pi}{16}$

$\theta_2 = \dfrac{13\pi}{16}$

and $\theta_3 = \dfrac{21\pi}{16} \left(\equiv \dfrac{-11\pi}{16}\right)$

Hence the four solutions are

$$\sqrt{6}e^{(\frac{-3\pi}{16})i},\ \sqrt{6}e^{(\frac{5\pi}{16})i},\ \sqrt{6}e^{(\frac{13\pi}{16})i},\ \text{and } \sqrt{6}e^{(\frac{-11\pi}{16})i}.$$

7. Let $P(x) = x^4 - 4x^3 + 8x^2 - 8x - 60$.

[4] (a) Apply Descartes' rule of signs to $P(x)$.

**Solution:**
Since there are 3 sign changes in $P(x)$, the number of positive real roots of $P(x)$ is 3 or 1.
$P(-x) = x^4 + 4x^3 + 8x^2 + 8x - 60$
Since $P(-x)$ has 1 sign change, the number of negative real roots of $P(x)$ is 1.

[8] (b) Given that $1 - 3i$ is a root of $P(x)$, express $P(x)$ as a product of real linear and real irreducible quadratic terms.

**Solution:** We know that since $1 - 3i$ is a root, then so $1 + 3i$ is also a root. Hence $(x - (1 - i))$ and $(x - (1 + i))$ are both factors.

So $(x - (1 - 3i))(x - (1 + 3i)) = (x^2 - 2x + 10)$ is also a factor.

$$\begin{array}{r|l} & x^2 - 2x - 6 \\ \hline x^2 - 2x + 10 & x^4 - 4x^3 + 8x^2 - 8x - 60 \\ & \underline{x^4 - 2x^3 + 10x^2} \\ & -2x^3 - 2x^2 - 8x \\ & \underline{-2x^3 + 4x^2 - 4x} \\ & -6x^2 + 12x - 60 \\ & \underline{-6x^2 + 12x - 60} \\ & 0 \end{array}$$

To find the roots of $x^2 - 2x - 6$ we use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, so

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-6)}}{2(1)} \\ &= \frac{2 \pm \sqrt{4 + 24}}{2} \\ &= \frac{2 \pm 2\sqrt{7}}{2} \\ &= 1 \pm \sqrt{7} \end{aligned}$$

(Note that these two roots are real, but irrational.)

Hence $P(x) = (x - (1 + \sqrt{7}))(x - (1 - \sqrt{7}))(x^2 - 2x + 10)$.