UNIVERSITY OF MANITOBA

DATE: June 9, 2009 MIDTERM II

TITLE PAGE

COURSE: MATH 1210 TIME: 50 minutes

EXAMINATION: Classical and Linear Algebra EXAMINER: M. Davidson

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FAMILY NAME: (Print in ink) ____________________

GIVEN NAME(S): (Print in ink) ____________________

STUDENT NUMBER: ____________

SIGNATURE: (in ink) ____________________

(I understand that cheating is a serious offense)

INSTRUCTIONS TO STUDENTS:

This is a 50 minute exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 5 pages of questions and also 1 blank page for rough work. Please check that you have all the pages. You may remove the blank page if you want, but be careful not to loosen the staple.

The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 50 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	6	
2	4	
3	8	
4	10	
5	10	
6	12	
Total:	50	

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[6] 1. For each of the following pairs $\overrightarrow{u}$, $\overrightarrow{v}$, find $\overrightarrow{u} \cdot \overrightarrow{v}$. If the vectors are orthogonal, indicate so by writing 'ORTHO', else write 'NOT ORTHO'.

(a) $\overrightarrow{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ $\overrightarrow{v} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$

Solution:
$\overrightarrow{u} \cdot \overrightarrow{v} = (1)(7) + (-3)(4) = 7 + (-12) = -5$
NOT ORTHO

(b) $\overrightarrow{u} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$ $\overrightarrow{v} = \begin{bmatrix} 7 \\ -4 \\ -2 \end{bmatrix}$

Solution:
$\overrightarrow{u} \cdot \overrightarrow{v} = (2)(7) + (5)(-4) + (-3)(-2) = 14 + (-20) + 6 = 0$
ORTHO

(c) $\overrightarrow{u} = \begin{bmatrix} 3 \\ -2 \\ -11 \end{bmatrix}$ $\overrightarrow{v} = \begin{bmatrix} 5 \\ 4 \\ -1 \end{bmatrix}$

Solution:
$\overrightarrow{u} \cdot \overrightarrow{v} = (3)(5) + (-2)(4) + (-11)(-1) = 15 + (-8) + 11 = 18$
NOT ORTHO

[4] 2. Find the determinant of the matrix $A = \begin{pmatrix} -1 & 3 & 4 \\ 1 & 6 & -5 \\ -2 & 2 & 7 \end{pmatrix}$ by expansion along the second column. Show your steps carefully.

Solution:

$$\begin{aligned}
\det A &= (3)(-1)^{1+2}\begin{vmatrix} 1 & -5 \\ -2 & 7 \end{vmatrix} + (6)(-1)^{2+2}\begin{vmatrix} -1 & 4 \\ -2 & 7 \end{vmatrix} + (2)(-1)^{3+2}\begin{vmatrix} -1 & 4 \\ 1 & -5 \end{vmatrix} \\
&= -3(7-10) + 6(-7+8) - 2(5-4) \\
&= -3(-3) + 6(1) - 2(1) \\
&= 9 + 6 - 2 \\
&= 13
\end{aligned}$$

[8] 3. Find, in standard form, an equation of the plane that contains the point $(3, 4, -1)$ and the line $\begin{bmatrix} x, & y, & z \end{bmatrix} = \begin{bmatrix} 1, & 3, & 2 \end{bmatrix} + t \begin{bmatrix} -1, & 1, & -1 \end{bmatrix}$.

(Hint: Can you use the information given to find two vectors in the plane? How would you use that information to find the plane normal?)

Solution:

Since the line is in the plane and $\begin{bmatrix} -1, & 1, & -1 \end{bmatrix}$ is a vector in the direction of the line, then $\begin{bmatrix} -1, & 1, & -1 \end{bmatrix}$ is a vector in the plane. Also, since $(3, 4, -1)$ and $(1, 3, 2)$ are points in the plane (the second point comes from the line at $t = 0$), then the vector $\overrightarrow{(3, 4, -1)(1, 3, 2)} = [1 - 3, 3 - 4, 2 + 1] = [-2, -1, 3]$ is also a vector in the plane.

The cross product of two vectors in the plane will give a vector normal to the plane (called $\overrightarrow{n}$).

Hence

$$\begin{aligned}
\overrightarrow{n} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & -1 \\ -2 & -1 & 3 \end{vmatrix} \\
&= \hat{\imath} \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} -1 & -1 \\ -2 & 3 \end{vmatrix} + \hat{k} \begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix} \\
&= (3 - 1)\hat{\imath} - (-3 - 2)\hat{\jmath} + (1 + 2)\hat{k} \\
&= 2\hat{\imath} + 5\hat{\jmath} + 3\hat{k} \\
&= [2, 5, 3]
\end{aligned}$$

Hence the plane is described by:

$$\begin{aligned}
[2, 5, 3] \cdot [x - 3, y - 4, z + 1] &= 0 \\
2(x - 3) + 5(y - 4) + 3(z + 1) &= 0 \\
2x - 6 + 5y - 20 + 3z + 3 &= 0 \\
2x + 5y + 3z &= 23
\end{aligned}$$

The standard form of the plane is $2x + 5y + 3z = 23$.

[10] 4. Let $A = \begin{pmatrix} 4 & -1 & 5 \\ 3 & 0 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -1 \\ 2 & 0 \\ -3 & 5 \end{pmatrix}$, $C = \begin{pmatrix} 3 & -4 \\ 2 & 7 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 3 & -5 \\ 1 & 0 & -1 \\ -4 & 2 & 5 \end{pmatrix}$,

$E = \begin{pmatrix} -1 & 3 & 2 \end{pmatrix}$ and $F = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}$.

Evaluate the expression if it is defined. If it is undefined, *clearly* explain why.

(a) $AB + C + I_2$

Solution:

$$\begin{aligned} AB + C + I_2 &= \begin{pmatrix} 4 & -1 & 5 \\ 3 & 0 & 6 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 2 & 0 \\ -3 & 5 \end{pmatrix} + \begin{pmatrix} 3 & -4 \\ 2 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -5 & 21 \\ -9 & 27 \end{pmatrix} + \begin{pmatrix} 3 & -4 \\ 2 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 17 \\ -7 & 35 \end{pmatrix} \end{aligned}$$

(b) $BC + AD$

Solution: Undefined

Since B is a 3×2 and C is a 2×2, then BC is a 3×2 matrix. Since A is a 2×3 and D is a 3×3, then AD is a 2×3 matrix. Hence AD cannot be added to BC, since you cannot add matrices of different sizes.

(c) $2E^T - 3DF$

Solution:

$$\begin{aligned} 2E^T - 3DF &= 2\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 2 & 3 & -5 \\ 1 & 0 & -1 \\ -4 & 2 & 5 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix} - 3\begin{pmatrix} 31 \\ 8 \\ -31 \end{pmatrix} \\ &= \begin{pmatrix} -2 - 93 \\ 6 - 24 \\ 4 + 93 \end{pmatrix} = \begin{pmatrix} -95 \\ -18 \\ 97 \end{pmatrix} \end{aligned}$$

[10] 5. For each of the following matrices, do ALL of the following:

(a) Decide if the matrix is in row echelon form. If it is, write 'REF' beside or below, else write 'NOT REF.

(b) Decide if the matrix is in reduced row echelon form. If it is, write 'RREF' beside or below, else write 'NOT RREF.

(c) Interpret the matrix as row equivalent to the augmented matrix of a system of equations (having variables x_1, x_2, etc.). Find the solution to that system.

Solution:

$$A = \begin{pmatrix} 1 & 5 & 0 & 0 & 7 & 0 & 7 \\ 0 & 0 & 1 & 0 & -2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \end{pmatrix} \quad \begin{matrix}\text{REF} \\ \text{RREF}\end{matrix}$$

Since the columns associated with x_2 and x_5 have no leading ones, we let $x_2 = s$ and $x_5 = t$, then the solutions are :

$$\begin{aligned} x_1 &= 7 - 5s - 7t \\ x_2 &= s \\ x_3 &= 4 + 2t \\ x_4 &= -2 - 3t \\ x_5 &= t \\ x_6 &= 6 \end{aligned}$$

$$B = \begin{pmatrix} 1 & -3 & 4 & 4 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & -3 \end{pmatrix} \quad \begin{matrix}\text{REF} \\ \text{NOT RREF}\end{matrix}$$

Using back substitution:

$$x_3 = -3$$

$$\begin{aligned} x_2 - 2x_3 &= -4 \\ x_2 - 2(-3) &= -4 \\ x_2 &= -10 \end{aligned}$$

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 4 \\ x_1 - 3(-10) + 4(-3) &= 4 \\ x_1 &= -14 \end{aligned}$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{matrix}\text{NOT REF} \\ \text{NOT RREF}\end{matrix}$$

Note the bottom row implies $0 = 1$. Hence this system has no solutions.

[12] 6. Solve the following system using Gauss-Jordan elimination:

$$\begin{array}{rcrcrl} 3x & + & 5y & + & z = & 5 \\ 2x & + & 3y & + & 2z = & 8 \\ 4x & + & 10y & - & z = & 4 \end{array}$$

Solution:

$$\left(\begin{array}{rrr|r} 3 & 5 & 1 & 5 \\ 2 & 3 & 2 & 8 \\ 4 & 10 & -1 & 4 \end{array}\right) \Longrightarrow R_1 \rightarrow R_1 - R_2$$

$$\left(\begin{array}{rrr|r} 1 & 2 & -1 & -3 \\ 2 & 3 & 2 & 8 \\ 4 & 10 & -1 & 4 \end{array}\right) \Longrightarrow \begin{array}{lcl} R_2 & \rightarrow & R_2 - 2R_1 \\ R_3 & \rightarrow & R_3 - 4R_1 \end{array}$$

$$\left(\begin{array}{rrr|r} 1 & 2 & -1 & -3 \\ 0 & -1 & 4 & 14 \\ 0 & 2 & 3 & 16 \end{array}\right) \Longrightarrow R_2 \rightarrow -R_2$$

$$\left(\begin{array}{rrr|r} 1 & 2 & -1 & -3 \\ 0 & 1 & -4 & -14 \\ 0 & 2 & 3 & 16 \end{array}\right) \Longrightarrow \begin{array}{lcl} R_1 & \rightarrow & R_1 - 2R_2 \\ R_3 & \rightarrow & R_3 - 2R_2 \end{array}$$

$$\left(\begin{array}{rrr|r} 1 & 0 & 7 & 25 \\ 0 & 1 & -4 & -14 \\ 0 & 0 & 11 & 44 \end{array}\right) \Longrightarrow R_3 \rightarrow (\tfrac{1}{11})R_3$$

$$\left(\begin{array}{rrr|r} 1 & 0 & 7 & 25 \\ 0 & 1 & -4 & -14 \\ 0 & 0 & 1 & 4 \end{array}\right) \Longrightarrow \begin{array}{lcl} R_1 & \rightarrow & R_1 - 7R_3 \\ R_2 & \rightarrow & R_2 + 4R_3 \end{array}$$

$$\left(\begin{array}{rrr|r} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right) \qquad \begin{array}{lcr} x & = & -3 \\ y & = & 2 \\ z & = & 4 \end{array}$$