UNIVERSITY OF MANITOBA

DATE: April 20, 2009 — FINAL EXAMINATION

PAPER # 523 — TITLE PAGE

EXAMINATION: Techniques of Classical and Linear Algebra — TIME: 2 hours

COURSE: MATH 1210 — EXAMINER: Berry, Borgersen

NAME: (Print in ink) ______________________________

STUDENT NUMBER: ______________

SEAT NUMBER: ______________

SIGNATURE: (in ink) ______________________________

(I understand that cheating is a serious offense)

Please indicate your section / instructor

- ❑ A01 Dr. T. Berry
- ❑ A02 R. Borgersen

INSTRUCTIONS TO STUDENTS:

This is a 2 hour exam. **Please show your work clearly.**

You may have in your possession one information page (8.5" by 11", handwritten, one-sided, your name and ID# must be clearly marked). **If your sheet does not conform to these criteria, it will be confiscated, and you will be charged with bringing unauthorized material into a test.**

There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 10 pages, including 2 pages for rough work. Please check that you have all the pages. You may remove the blank pages if you want, but be careful not to loosen the staple.

The value of each question is indicated in the left hand margin beside the statement of the question. The total value of all questions is 100 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	12	
2	6	
3	7	
4	7	
5	8	
6	10	
7	6	
8	6	
9	4	
10	10	
11	12	
12	12	
Total:	100	

---

[12] 1. Given a square matrix A, the powers of A (denoted by A^n, $n > 0$) are defined recursively by $A^2 = A \cdot A$ and $A^n = A \cdot A^{n-1}$ for $n \geq 3$.

Suppose that for some $x \in \mathbb{R}$, $A = \begin{bmatrix} 1 & x \\ 0 & x \end{bmatrix}$.

(a) Calculate A^2 and A^3.

(b) Use induction to show that for all $n \geq 1$,

$$A^n = \begin{bmatrix} 1 & \displaystyle\sum_{\ell=1}^{n} x^\ell \\ 0 & x^n \end{bmatrix}.$$

---

[6] 2. Let z_1 and z_2 be two complex numbers, z_1 having argument θ_1 and z_2 having argument θ_2. Find the argument of

$$\frac{z_1^4}{\overline{z_2}^2}.$$

[7] 3. Let $p(x) = x^4 - x^3 + 6x^2 + 14x - 20$. Given that $p(1-3i) = 0$, find all the zeros of $p(x)$.

---

[7] 4. Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}.$$

Find $(BA)^T$.

[8] 5. Suppose that the matrix

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -3 & 0 & 3 \\ 0 & 1 & 1 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

is the RREF of the augmented matrix of a system of linear equations $Ax = b$.

(a) How many unknowns are in the system?

(b) What is the rank of the coefficient matrix of the system?

(c) How many "free" variables (parameters) occur in the solutions of the system?

(d) Find all solutions identifying clearly which variables are the "free" variables.

---

[10] 6. Consider the two planes

$$\Pi_1 : x + 2y + 3z = 6$$

$$\Pi_2 : -2x + y + z = 0.$$

(a) Use Gauss-Jordan elimination to find the line of intersection of these two planes. Write your answer in parametric form.

(b) Find a vector along the line of intersection.

(c) Show that the vector found in part (b) is perpendicular to the normal vectors of both planes.

---

[6] 7. Consider the linear system

$$\begin{array}{rcrcrcl} & & x_2 & + & x_3 & = & 2 \\ 5x_1 & + & x_2 & - & x_3 & = & 3 \\ 2x_1 & - & 3x_2 & - & 3x_3 & = & 6 \end{array}$$

(a) Justify the claim that this system has a unique solution.

(b) Find x_2.

[6] 8. Consider the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}.$$

(a) Give one reason why you know this set is linearly dependent (No work required).

(b) Give one reason why at least one of the vectors $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ may be expressed as a non-trivial linear combination of the others.

(c) Find a subset of these vectors (containing as many vectors as possible) that forms a linearly independent set.

---

[4] 9. Let A be any 3×3 matrix such that $A^T = -A$. Show that A is non-invertible (*i.e.*, singular).

[10] 10. Let

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & -3 & 0 & 2 \end{bmatrix}.$$

(a) Evaluate $\det(A)$ (Show all your work and briefly describe the method being used).

(b) Find the missing entry in

$$\mathrm{adj}(A) = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 6 & 0 & -3 \\ 2 & 0 & -1 & 0 \\ 0 & \underline{\hspace{2em}} & 0 & -6 \end{bmatrix}.$$

(c) Find A^{-1}.

---

[12] 11. Consider the linear transformation in $\mathbb{E}^3$ given by $v = Au$ where

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

(a) Describe the effect of the transformation on the unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$. Write your answers as linear combinations of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

(b) Show that A is invertible.

(c) Use the results of part (a) to determine the effect of the transformation $v = A^{-1}u$ on the unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$.

(d) Find A^{-1} using the information in part (c).

---

[12] 12. Consider the linear transformation $v = Au$ with

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) Find the eigenvectors of A corresponding to the eigenvalue having multiplicity 2.

(c) Express your answer in part (b) as a linear combination of two linearly independent vectors.

SCRAP PAPER

SCRAP PAPER