Mathematics 1210 Assignment 1 Winter 2012

Hand in on January 20 as instructed by your section lecturer. See the "Template for Induction" on the course webpage for induction questions. Use the HONESTY DECLARATION (see link on course web page) as a cover page for all of your assignments. It is recommended that you write only on one side and attach pages with a single staple in the top left corner, without plastic covers, paper clips or other fasteners.

1. Use mathematical induction to prove that, for all $n \geq 1$,

$$\sum_{i=1}^{n}(2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}.$$

Sol: For $n \geq 1$, let P_n be the assertion that $\displaystyle\sum_{i=1}^{n}(2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}$.

P_1 says $\displaystyle\sum_{i=1}^{1}(2i-1)^2 = \frac{1(2\cdot 1-1)(2\cdot 1+1)}{3}$. That is, $1^2 = 1$, which is true.

Let $k \geq 1$, and assume that P_k is true: $\displaystyle\sum_{i=1}^{k}(2i-1)^2 = \frac{k(2k-1)(2k+1)}{3}$.

We wish to show that P_{k+1} follows: $\displaystyle\sum_{i=1}^{k+1}(2i-1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$.

Now,

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i-1)^2 &= \sum_{i=1}^{k}(2i-1)^2 + (2k+1)^2 \\
&= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \qquad \text{(By inductive hypothesis)} \\
&= (2k+1)\frac{k(2k-1)+3(2k+1)}{3} \\
&= \frac{(2k+1)(2k^2+5k+3)}{3} \\
&= \frac{(2k+1)(k+1)(2k+3)}{3}
\end{aligned}$$

Thus, P_k implies P_{k+1}.

Therefore, by induction, P_n is true for all $n \geq 1$. $\square$

2. Use mathematical induction to prove that, for all $n \geq 1$,

$$1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^{2n}} = \frac{4^{2n+1} - 1}{3 \cdot 4^{2n}}.$$

Sol: For $n \geq 1$, let P_n be the assertion that $1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^{2n}} = \frac{4^{2n+1}-1}{3 \cdot 4^{2n}}$.

P_1 says $1 + \frac{1}{4} + \frac{1}{4^2} = \frac{4^{2\cdot 1+1}-1}{3 \cdot 4^{2\cdot 1}}$. That is, $\frac{21}{16} = \frac{63}{48}$, which is true.

Let $k \geq 1$, and assume that P_k is true: $1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^{2k}} = \frac{4^{2k+1}-1}{3 \cdot 4^{2k}}$.

We wish to show that P_{k+1} follows: $1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^{2k+2}} = \frac{4^{2k+3}-1}{3 \cdot 4^{2k+2}}$.

Now,

$$\begin{aligned}
LHS &= 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^{2k+2}} \\
&= \left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^{2k}}\right) + \frac{1}{4^{2k+1}} + \frac{1}{4^{2k+2}} \\
&= \frac{4^{2k+1} - 1}{3 \cdot 4^{2k}} + \frac{1}{4^{2k+1}} + \frac{1}{4^{2k+2}} && \text{(By inductive hypothesis)} \\
&= \frac{4^2\left(4^{2k+1} - 1\right) + 12 + 3}{3 \cdot 4^{2k+2}} \\
&= \frac{4^{2k+3} + 1}{3 \cdot 4^{2k+2}} = RHS
\end{aligned}$$

$\therefore P_k \Rightarrow P_{k+1}$.

Therefore, by induction, P_n is true for all $n \geq 1$. □

3. (a) Find the value of $\sum_{i=1}^{n} \frac{2}{i(i+2)}$ (HINT: Each term is a difference of simple fractions, giving a sum that "telescopes".)

(b) Use mathematical induction to prove that your answer to part (a) is correct.

Sol: (a) For $i \geq 1$, $\frac{2}{i(i+2)} = \frac{1}{i} - \frac{1}{i+2}$, so

$$\begin{aligned}
\sum_{i=1}^{n} \frac{2}{i(i+2)} &= \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+2} \\
&= (\frac{1}{1} - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + \cdots + (\frac{1}{n-1} - \frac{1}{n+1}) + (\frac{1}{n} - \frac{1}{n+2}) \\
&= 1 + \frac{1}{2} + (\frac{1}{3} - \frac{1}{3}) + (\frac{1}{4} - \frac{1}{4}) + \cdots + (\frac{1}{n} - \frac{1}{n}) - \frac{1}{n+1} - \frac{1}{n+2} \\
&= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \\
&= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{2(n+1)(n+2)} \\
&= \frac{3n^2 + 5n}{2(n+1)(n+2)}
\end{aligned}$$

(or $\frac{n(3n+5)}{2(n+1)(n+2)}$, or $\frac{n(3n+5)}{2n^2+6n+4}$, etc.)

Alternatively, one can do the cancellation by working in sigma notation and doing a change of index:

$$\begin{aligned}
\sum_{i=1}^{n} \frac{2}{i(i+2)} &= \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+2} \\
&= \sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{n} \frac{1}{i+2} \\
&= 1 + \frac{1}{2} + \sum_{i=3}^{n} \frac{1}{i} - \left(\sum_{i=3}^{n} \frac{1}{i} + \frac{1}{n+1} + \frac{1}{n+2} \right) \\
&= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \\
&= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{2(n+1)(n+2)} \\
&= \frac{3n^2 + 5n}{2(n+1)(n+2)}
\end{aligned}$$

Sol: (b) For $n \geq 1$, let P_n be the assertion that $\displaystyle\sum_{i=1}^{n} \frac{2}{i(i+2)} = \frac{3n^2+5n}{2(n+1)(n+2)}$.

P_1 says $\displaystyle\sum_{i=1}^{1} \frac{2}{i(i+2)} = \frac{3 \cdot 1^2 + 5 \cdot 1}{2(1+1)(1+2)}$. That is, $\frac{2}{3} = \frac{8}{12}$, which is true.

Let $k \geq 1$, and assume that P_k is true: $\displaystyle\sum_{i=1}^{k} \frac{2}{i(i+2)} = \frac{3k^2+5k}{2(k+1)(k+2)}$.

We wish to show that P_{k+1} follows:

$$\sum_{i=1}^{k+1} \frac{2}{i(i+2)} = \frac{3(k+1)^2+5(k+1)}{2(k+2)(k+3)} = \frac{3k^2+11k+8}{2(k+2)(k+3)}.$$

Now,

$$\begin{aligned}
LHS &= \sum_{i=1}^{k+1} \frac{2}{i(i+2)} \\
&= \left(\sum_{i=1}^{k} \frac{2}{i(i+2)} \right) + \frac{2}{(k+1)(k+3)} \\
&= \frac{3k^2+5k}{2(k+1)(k+2)} + \frac{2}{(k+1)(k+3)} && \text{(By induction hypothesis)} \\
&= \frac{1}{k+1} \left(\frac{3k^2+5k}{2(k+2)} + \frac{2}{k+3} \right) \\
&= \frac{1}{k+1} \frac{(3k^2+5k)(k+3)+2(2k+4)}{2(k+2)(k+3)} \\
&= \frac{3k^3+14k^2+19k+8}{2(k+1)(k+2)(k+3)},
\end{aligned}$$

whereas

$$RHS = \frac{(3k^2+11k+8)(x+1)}{2(k+2)(k+3)(x+1)} = \frac{3k^3+14k^2+19k+8}{2(k+1)(k+2)(k+3)} (= LHS).$$

Thus, P_k implies P_{k+1}.

Therefore, by induction, P_n is true for all $n \geq 1$. $\square$

4. Use mathematical induction to prove that, for any $n \geq 0$, the polynomial $x^{2n+1} + 1$ is divisible by $x + 1$.

Sol: For any $n \geq 1$, let P_n be the statement that $(x+1) | \left(x^{2n+1} + 1\right)$.

P_1 says that $(x+1)|(x^3+1)$. Since $x^3 + 1 = (x+1)(x^2+x+1)$, this is true.

Now suppose $k \geq 1$ and that P_k is true. That is,

$$(x+1)| \left(x^{2k+1} + 1\right)$$

We wish to show P_{k+1}. That is, $(x+1)| \left(x^{2k+3} + 1\right)$.

By the induction hypothesis, $x + 1$ divides $x^{2k+1} + 1$.

Further, we see that $x + 1$ divides $(x^{2k+3} + 1) - (x^{2k+1} + 1) = x^{2k+3} - x^{2k+1} = x^{2k+1}\left(x^2 - 1\right) = (x-1)x^{2k+1}(x+1)$.

It follows that $x+1$ divides $x^{2k+1} + 1 + \left[(x^{2k+3}+1) - (x^{2k+1}+1)\right] = (x^{2k+3}+1)$.

Thus, P_k implies P_{k+1}

So by mathematical induction, P_n is true for all $n \geq 1$.

5. Use mathematical induction to prove that, for all $n \geq 1$,

$$2^{n+1} > n^2.$$

(HINT: You may find it helpful to take $n = 3$ as a base case.)

Sol: For all $n \geq 1$ let P_n be the statement that $2^{n+1} > n^2$.

P_1 says $2^2 > 1^2$, or $4 > 1$, which is true.
P_2 says $2^3 > 2^2$, or $8 > 4$, which is true.
P_3 says $2^4 > 3^2$, or $16 > 9$, which is true (Note: this is our base case).

Now suppose $k \geq 3$ and P_k is true. That is, $2^{k+1} > k^2$.

We wish to show that P_{k+1} is also true. That is, $2^{k+2} > (k+1)^2$.

By the induction hypothesis we have

$$LHS = 2^{k+2} = 2 \cdot 2^{k+1} > 2k^2 = k^2 + k^2.$$

Further,

$$RHS = (k+1)^2 = k^2 + 2k + 1.$$

Now, for $k \geq 3$, $k^2 \geq 3k \geq 2k + 1$, so $k^2 + k^2 > k^2 + 2k + 1$. It follows that $2^{k+2} > (k+1)^2$.
$\therefore, P_k \Rightarrow P_{k+1}$.
It follows by mathematical induction that P_n is true for all $n \geq 3$ and so (since P_1, P_2 are true) for all $n \geq 1$.

NOTE: It is possible to use different base cases. For example, this inductive step uses base case $n = 1$ (it's a bit trickier and as we show it, the proof is not quite complete):

$$LHS = 2^{k+2} = 2 \cdot 2^{k+1} = 2^{k+1} + 2^{k+1} > k^2 + 2^{k+1}$$

(Induction hyptothesis used for the final inquality) Further,

$$RHS = (k+1)^2 = k^2 + 2k + 1.$$

Since $2^{k+1} = 2 \cdot 2^k$ and $2^k > k$ (we'll accept this without proof, though it really is only "proof by assertion!", which is not a valid argument), $2^{k+1} > 2k$, so $2^{k+1} \geq 2k + 1$ (since it is an integer). So we have

$$LHS > k^2 + 2^{k+1} \geq k^2 + 2k + 1 = (k+1)^2 = RHS.$$

6. (a) Write $\sum_{k=1}^{m}(2+3k)(4-k)$ as a sum whose terms are themselves sums of the form $c\sum k^a$, where c is some number and a may be any of $0, 1$ or 2.

(b) Use the formulas $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ to find the value of the sum in part (a).

(c) Rewrite the sum $\sum_{i=12}^{111} 2^{i-6}(i+9)$ (i.e., by a change of index) so that the resulting initial and terminal index values are 1 and 100.

Sol: (a) $\sum_{k=1}^{m} 8 + 10\sum_{k=1}^{m} k - 3\sum_{k=1}^{m} k^2$

(b) $\sum_{k=1}^{m} 8 + 10\sum_{k=1}^{m} k - 3\sum_{k=1}^{m} k^2 = 8m + 10\frac{m(m+1)}{2} - 3\frac{m(m+1)(2m+1)}{6}$

$$= m\left[8 + \frac{(m+1)(10-(2m+1))}{2}\right] = m\left[\frac{16+(m+1)(9-2m)}{2}\right]$$

$$= \frac{m(25+7m-2m^2)}{2}$$

(c) $\sum_{i=1}^{100} 2^{i+5}(i+20)$