Math 1210 Assignment 3 Winter 2012

Due Friday March 9; hand in as instructed in class. Same instructions as in Assignment #1. Note: one staple in the top left corner is required. This problem set reprises old material on vectors and geometry because this was poorly done on the midterm.

1. Given points $A(8, 1, -2)$, $B(7, -1, -9)$ and $C(6, -1, -5)$

   (a) Point D is the midpoint of the line segment $\overline{AB}$. Find the coordinates of D.

   (b) Find the coordinates of point E, which is $\frac{1}{3}$ of the way from A to B on segment $\overline{AB}$.

   (c) Find the lengths of all three sides of triangle $\triangle ABC$. What special kind of triangle is this?

Sol: (a) $\overrightarrow{OD} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = [8, 1, -2] + \frac{1}{2}[-1, -2, -7] = [\frac{15}{2}, 0, -\frac{11}{2}]$

(b) $\overrightarrow{OD} = \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB} = [8, 1, -2] + \frac{1}{3}[-1, -2, -7] = [\frac{23}{3}, \frac{1}{3}, -\frac{13}{3}]$

(c) $||\overrightarrow{AB}|| = ||\ [-1, -2, -7]\ | = \sqrt{1 + 4 + 49} = \sqrt{54} = 3\sqrt{6}$

$||\overrightarrow{BC}|| = ||\ [-1, 0, 4]\ || = \sqrt{1 + 16} = \sqrt{17}$

$||\overrightarrow{CA}|| = ||\ [2, 2, 3]\ || = \sqrt{4 + 4 + 9} = \sqrt{17}$

This triangle is isosceles (since $||\overrightarrow{BC}|| = ||\overrightarrow{CA}||$)

2. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be unit vectors. Using everything we know about vectors, simplify the following vector expression as far as you can:

$$((\mathbf{u} \times \mathbf{v}) + \mathbf{u}) \cdot (\mathbf{u} + \mathbf{v} + 6\mathbf{w}) - (\mathbf{v} + 2\mathbf{w}) \cdot (3\mathbf{u} + \mathbf{v})$$

Sol:

$$((\mathbf{u} \times \mathbf{v}) + \mathbf{u}) \cdot (\mathbf{u} + \mathbf{v} + 6\mathbf{w}) - (\mathbf{v} + 2\mathbf{w}) \cdot (3\mathbf{u} + \mathbf{v})$$

$$\begin{aligned}
&= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} + (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} + 6(\mathbf{u} \times \mathbf{v}) + \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + 6\mathbf{u} \cdot \mathbf{w} \\
&\quad - 3\mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} - 6\mathbf{w} \cdot \mathbf{u} - 2\mathbf{w} \cdot \mathbf{v} \\
&= \mathbf{0} + \mathbf{0} + 6(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} + ||\mathbf{u}||^2 - ||\mathbf{v}||^2 + (1-2)\mathbf{u} \cdot \mathbf{v} + (6-6)\mathbf{u} \cdot \mathbf{w} - 2\mathbf{v} \cdot \mathbf{w} \\
&= 6(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} + 1 - 1 - \mathbf{u} \cdot \mathbf{v} + \mathbf{0} - 2\mathbf{v} \cdot \mathbf{w} \\
&= 6(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} - \mathbf{u} \cdot \mathbf{v} - 2\mathbf{v} \cdot \mathbf{w}
\end{aligned}$$

3. Lines and planes:

(a) Express the line obtained as the intersection of planes $x + 2y - 5z = 10$ and $2x + y - 3z = 20$, in parametric form.

(b) Find an equation in standard form for the plane perpendicular to the line $\mathbf{v} = [1, 3, 5] + t[0, 1, 2]$ and passing through the point $P_0(1, 1, 1)$

(c) At which point does line $\mathbf{v} = [3, 2, 1] + t[2, 2, -1]$ meet the plane $x - 2y + 3z = 1$?

Sol: (a) Subtracting twice the first equation from the second yields the system of equations

$$x + 2y - 5z = 10; \qquad -3y + 7z = 0$$

z is free, so we equate it to a parameter t: $z = t$.

Then $y = \frac{7}{3}z = \frac{7}{3}t$ and $x = 10 - 2y + 5z = 10 - \frac{14}{3}t + 5t = 10 + \frac{1}{3}t$.

The intersection of the planes is the solution of this system, that is $[x, y, z] = \left[10 + \frac{1}{3}t, \frac{7}{3}t, t\right]$, or

$$[x, y, z] = [10, 0, 0] + \left[\frac{1}{3}, \frac{7}{3}, 1\right] t$$

which is the parametric form for the required line.

[NOTES: column vectors may also be used. The direction vector must be a multiple of $[1, 7, 3]$. The position vector may be for any point on the line instead of $[10, 0, 0]$. Another approach: to take cross products of the normal vectors of the two planes for a direction vector, but you must still produce a position vector. $[10, 0, 0]$ may be found by inspection, or something like a solution to the system must probably be attempted]

(b) Take the direction vector of the line to be a normal vector to the plane, and $[1, 1, 1]$ is a point in the plane. So the point-normal form of the plane is

$$[0, 1, 2] \cdot ([x, y, z] - [1, 1, 1]) = 0.$$

Simplifying gives $y + 2z - 3 = 0$, or, in the required standard form, $y + 2z = 3$

[NOTE: $y + 2z - 3 = 0$ or any multiple of either form is okay.]

(c) Substituting the components of $\mathbf{v} = [x, y, z] = [3+2t, 2+2t, 1-t]$ into the equation of the line gives

$$(3 + 2t) - 2(2 + 2t) + 3(1 - t) = (3 - 4 + 3) + (2 - 4 - 3)t = 2 - 5t = 1,$$

so $t = \frac{1}{5}$ and the required point has position vector $[3, 2, 1] + \frac{1}{5}[2, 2, -1] = \left[\frac{17}{5}, \frac{12}{5}, \frac{4}{5}\right]$.

4. For the system of equations

$$\begin{array}{rcrcrcrcl} & & v & - & x & - & y & = & 2 \\ -u & + & v & - & 2x & - & 3y & = & 3 \\ u & + & 2v & - & x & & & = & 3 \end{array}$$

(a) Express the system as a single matrix equation;

(b) Represent the system in the form of an augmented matrix;

(c) Put the system in REF using elementary row operations (annotate every step as shown in class);

(d) Solve the system.

Sol: (a) $\begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$

(b) $\left(\begin{array}{rrrr|r} 0 & 1 & -1 & -1 & 2 \\ -1 & 1 & -2 & -3 & 3 \\ 1 & 2 & -1 & 0 & 3 \end{array}\right)$

(c) Gaussian elimination not required, but a single ERO in every step is required (except when clearing above and below a leading 1). One possible such series:

$$\begin{aligned}
\left(\begin{array}{rrrr|r} 0 & 1 & -1 & -1 & 2 \\ -1 & 1 & -2 & -3 & 3 \\ 1 & 2 & -1 & 0 & 3 \end{array}\right)\begin{array}{l} R_3 \\ \\ R_1 \end{array} &\equiv \left(\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 3 \\ -1 & 1 & -2 & -3 & 3 \\ 0 & 1 & -1 & -1 & 2 \end{array}\right) R_2 + R_1 \\
&\equiv \left(\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 3 \\ 0 & 3 & -3 & -3 & 6 \\ 0 & 1 & -1 & -1 & 2 \end{array}\right) \tfrac{1}{3}R_2 \\
&\equiv \left(\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 3 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 1 & -1 & -1 & 2 \end{array}\right)\begin{array}{l} R_1 - 2R_2 \\ \\ R_3 - R_2 \end{array} \\
&\equiv \left(\begin{array}{rrrr|r} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).
\end{aligned}$$

[NOTE: Annotation here is done in the style of Section A02. Either section's style may be used but *no other*.]
(d) The matrix in RREF corresponds to the system of equations

$$\begin{aligned} u + x + 2y &= -1 \\ v - x - y &= 2 \\ 0 &= 0 \end{aligned}$$

The free variables are x, y. Setting $x = s$ and $y = t$ gives solution

$$\begin{bmatrix} u \\ v \\ x \\ y \end{bmatrix} = \begin{bmatrix} -1 - s - t \\ 2 + s + t \\ s \\ t \end{bmatrix}$$

[NOTE: The solution vector may be written as a row rather than a column, and with round or square brackets (or as four equations)]

5. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ a & b & c \end{pmatrix}$, where a, b, c are arbitrary.

(a) Which of the following can be obtained from A with a *single* Elementary Row Operation? (Answers only)

$$B_1 = A; \quad B_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2a+1 & 2b+2 & 2c+3 \end{pmatrix}; \quad B_3 = \begin{pmatrix} -1 & -3 & -5 \\ 1 & 2 & 3 \\ a & b & c \end{pmatrix}$$

(b) Which, if any, of the following can be obtained from A with a sequence of elementary row operations? (Answers only)

$$C_1 = -A; \quad C_2 = \begin{pmatrix} 1+a & 1+b & 1+c \\ 2 & 3 & 4 \\ a-1 & b-2 & c-3 \end{pmatrix}; \quad C_3 = \begin{pmatrix} 1+a & 2+b & 3+c \\ 1+a & 1+b & 1+c \\ 2 & 3 & 4 \end{pmatrix}$$

(c) Show how rows 2 and 3 of the matrix A can be interchanged using a sequence of row operations of *only* the second and third types (multiply a row by a nonzero constant or add a multiple of one row to another row).

Sol: (a) B_1 (multiply any row of A by 1 or add 0 times any row to any other row);
NOT B_2 (This is only possible with a series of *two or more* EROs);
B_3 (add -2 times R_2 to R_1)

(b) C_1 (multiply each of the rows of A by -1);
NOT C_2 (Observe that C_2 has rank 2 whereas A has rank 1, in general);
C_3 (Since row operations can be reversed by other row operations, it is easy to see how to get from C_3 to A: subtract rows 1 and 2 from row 3 and divide by -2 to obtain row $[a, b, c]$. Subtract this from the first two rows and then swap the first two rows.)

(c)

$$\begin{aligned}
A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ a & b & c \end{pmatrix} R_3 + R_2 \quad &\equiv \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1+a & 2+b & 3+c \end{pmatrix} R_2 - R_3 \\
&\equiv \begin{pmatrix} 1 & 1 & 1 \\ -a & -b & -c \\ 1+a & 2+b & 3+c \end{pmatrix} R_3 + R_2 \\
&\equiv \begin{pmatrix} 1 & 1 & 1 \\ -a & -b & -c \\ 1 & 2 & 3 \end{pmatrix} -R_3 \\
&\equiv \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 2 & 3 \end{pmatrix}
\end{aligned}$$