Math 1210 Assignment 4 Winter 2012
(UPDATED VERSION!)

Due Friday March 23; hand in as instructed in class. Same instructions as in Assignment #1. One staple in the top left corner is required.

1. Suppose A and B are square 2×2 matrices and $\det(B) \neq 0$. One of the following statements is true and the other is false (in general). Which is which? Justify the true one by mentioning two facts from the course; provide a counterexample for the other one.

   (a) $\det(AB^{-1}) = \frac{\det(A)}{\det(B)}$;

   (b) $\det(A - B) = \det(A) - \det(B)$.

Sol: The first statement is true because of the multiplicative property of determinants and that the determinant of an inverse of a matrix is the reciprocal of the determinant of the matrix. The first fact is Theorem 3.8; the second follows from it by noting

$$\det B \det B^{-1} = \det(BB^{-1}) = \det I = 1.$$

The second statement is false, as can be seen by taking $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\det(A - B) = -1$ whereas $\det(A) - \det(B) = 0 - 0 = 0$.

2. Use a series of cofactor expansions along strategically chosen rows/columns to obtain determinant $|A|$, and use a series of elementary row and/or column operations to obtain $|B|$, where:

$$|A| = \begin{vmatrix} 1 & 2 & 1 & 2 \\ -1 & 0 & 2 & 0 \\ 1 & 3 & 0 & 0 \\ 3 & 0 & -1 & 0 \end{vmatrix} \; ; \qquad |B| = \begin{vmatrix} 1 & 2 & 3 & 4 \\ -4 & 3 & -2 & 1 \\ 3 & 1 & 2 & 4 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

Sol: For the first one: - we first expand down the fourth column
- then down the second column
to obtain

$$|A| = \begin{vmatrix} 1 & 2 & 1 & 2 \\ -1 & 0 & 2 & 0 \\ 1 & 3 & 0 & 0 \\ 3 & 0 & -1 & 0 \end{vmatrix} = (-2)\begin{vmatrix} -1 & 0 & 2 \\ 1 & 3 & 0 \\ 3 & 0 & -1 \end{vmatrix} = (-2)(3)\begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} = (-6)(1-6) = 30.$$

For the second determinant, we subtract the first column from the others and expand along the last row to obtain

$$|B| = \begin{vmatrix} 1 & 2 & 3 & 4 \\ -4 & 3 & -2 & 1 \\ 3 & 1 & 2 & 4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 & 3 \\ -4 & 7 & 2 & 5 \\ 3 & -2 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 2 & 3 \\ 7 & 2 & 5 \\ -2 & -1 & 1 \end{vmatrix}.$$

Now adding the third row twice to each of the other rows and expanding down the second column gives:

$$|B| = (-1)\begin{vmatrix} -3 & 0 & 5 \\ 3 & 0 & 7 \\ -2 & -1 & 1 \end{vmatrix} = -(-1)(-1)\begin{vmatrix} -3 & 5 \\ 3 & 7 \end{vmatrix} = (-1)(-21-15) = 36.$$

NOTES: For $|A|$ the two columns used in the first two stages are the only "best" choices; others will lose marks. It does not matter how the resulting 2×2 minor is evaluated.

For $|B|$ there are many different sequences of row and/or column operations that may be used. However, you *must* use this method here or lose marks.

3. Solve the following system by following the steps of Gauss-Jordan elimination *exactly*:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & -2 \\ -x_1 & + & x_2 & + & 3x_3 & = & -1 \\ 3x_1 & - & 3x_2 & + & 2x_3 & = & -10 \end{array}$$

Sol: Gauss-Jordan elimination demands a particular series of row/column operations – but see the note below. You *must* annotate, as instructed in the course.

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ -1 & 1 & 3 & -1 \\ 3 & -3 & 2 & -10 \end{array}\right) \begin{array}{c} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array} = \left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & -1 & -4 \end{array}\right) \begin{array}{c} \\ \frac{1}{4}R_2 \\ \\ \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & -1 & -4 \end{array}\right) \begin{array}{c} R_1 - R_2 \\ \\ R_3 + R_2 \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & -1 & 0 & -\frac{5}{4} \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & -\frac{19}{4} \end{array}\right)$$

At this point one can stop and say that there are no solutions because there is a row of the form $(0|c)$ where $c \neq 0$.

NOTES:
(1) The textbook does not demand that the nonzero entry in the third column be the *first* nonzero entry (4) after the first row – so you are permitted to swap rows 2 and 3 then multiply row 2 by -1. Adding multiples of that row to the others you should arrive at the matrix $\left(\begin{array}{ccc|c} 1 & -1 & 0 & -6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -19 \end{array}\right)$.
(2) Aside from this alternative answer, there are no other possible matrices to reach using Gauss-Jordan Elimination as described in the text and class. However, students may proceed further so that the whole matrix is in RREF, though this is unnecessary. The conclusion that there are no solutions, however, is necessary.
(3) You must annotate in one of the two permitted styles.

4. Solve the following system using Cramer's Rule:

$$\begin{array}{rcrcrcl} & & -x_2 & + & 2x_3 & = & 2 \\ x_1 & & & + & 2x_3 & = & 4 \\ 2x_1 & + & x_2 & & & = & 6 \end{array}$$

Sol: The coefficient matrix is $A = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}$. Replacing each of the columns by the column of constant coefficients we obtain, respectively,

$$A_1 = \begin{pmatrix} 2 & -1 & 2 \\ 4 & 0 & 2 \\ 6 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 4 & 2 \\ 2 & 6 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 6 \end{pmatrix}.$$

The determinants of each of these are

$\det A = \begin{vmatrix} 0 & -1 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 2 \\ 1 & 0 & 2 \\ 2 & 0 & 2 \end{vmatrix} = -(-1)(2-4) = -2$,

$\det A_1 = \begin{vmatrix} 2 & -1 & 2 \\ 4 & 0 & 2 \\ 6 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 2 \\ 4 & 0 & 2 \\ 8 & 0 & 2 \end{vmatrix} = -(-1)(8-16) = -8$,

$\det A_2 = \begin{vmatrix} 0 & 2 & 2 \\ 1 & 4 & 2 \\ 2 & 6 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 2 \\ 1 & 4 & 2 \\ 0 & -2 & -4 \end{vmatrix} = -(1)(-8+4) = 4$, and

$\det A_3 = \begin{vmatrix} 0 & -1 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 2 \\ 1 & 0 & 4 \\ 2 & 0 & 8 \end{vmatrix} = -(-1)(8-8) = 0.$

So by Cramer's Rule the solution is

$[x_1, x_2, x_3] = \left[\frac{\det A_1}{\det A}, \frac{\det A_2}{\det A}, \frac{\det A_3}{\det A}\right] = \left[\frac{-8}{-2}, \frac{4}{-2}, \frac{0}{-2}\right] = [4, -2, 0].$

5. Find all values of x such that the following determinant is zero:

$$\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix}$$

Sol: The determinant (say by the basket-weaving formula) is

$$(x-1)^3+1+1-(x-1)-(x-1)-(x-1) = x^3-3x^2+3x-1+(5-3x) = x^3-3x^2+4$$

Now, $x = 2$ is a zero (you can find this either with the rational roots theorem or by observing that $x = 2$ gives a determinant with duplicate rows). Long division gives that the determinant is $(x-2)(x^2-x-2) = (x-2)^2(x+1)$, so the desired values are $x = 2$ and $x = -1$.

6. Let $\mathbf{x}_1$ be a solution to the non-homogeneous system whose matrix equation is $A\mathbf{x} = \mathbf{b}$, and let $\mathbf{x}_2$ be a solution to the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$. Prove that, for any $c \in \mathbb{R}$, $\mathbf{x}_1 + c\mathbf{x}_2$ is a solution to the first system, $A\mathbf{x} = \mathbf{b}$.

Sol: By assumption, $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{0}$. Taking $\mathbf{x} = \mathbf{x}_1 + c\mathbf{x}_2$, by simple matrix algebra we have

$$A\mathbf{x} = A(\mathbf{x}_1 + c\mathbf{x}_2) = A\mathbf{x}_1 + cA\mathbf{x}_2 = \mathbf{b} + c\mathbf{0} = \mathbf{b},$$

So $\mathbf{x} = \mathbf{x}_1 + c\mathbf{x}_2$ is a solution to the first system, as required.

7. Find a real (i.e., all entries real numbers) 2×2 matrix A that satisfies $A^2 = -I$. Use the properties of determinants to show that there are no real 2011×2011 matrices A which satisfy this equation.

Sol: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is one 2×2 matrix satisfying this equation.

Now 2011 is odd, so $|-I_{2011}| = (-1)^{2011}|I| = -1$ (by Theorem 3.6).

Whereas $|A^2| = |A|^2$ (by Theorem 3.8), which is positive.

So we cannot have $A^2 = -I_{2011}$.