Math 1210 Assignment 5 Winter 2012

Due Monday April 2; hand in as instructed in class.

Same instructions always. One staple in the top left corner is required.

1. (a) Express $\mathbf{x} = [5, 0, 3, 3]$ as a linear combination of vectors $\mathbf{u} = [1, 1, -1, 2]$, $\mathbf{v} = [3, 2, 1, 4]$, $\mathbf{w} = [2, 0, 2, 1]$.

   (b) Show that $\mathbf{y} = [1, 1, 1, 1]$ is not a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ (from in part 1a).

   (c) Show that every vector in $\mathbb{R}^3$ is a linear combination of vectors $\mathbf{a} = [-1, 1, 1]$, $\mathbf{b} = [1, -1, 1]$ and $\mathbf{c} = [1, 1, -1]$.

Sol:

(a)

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 1 & 2 & 0 & 0 \\ -1 & 1 & 2 & 3 \\ 2 & 4 & 1 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So

$$\mathbf{x} = 2\mathbf{u} - \mathbf{v} + 3\mathbf{w}.$$

(b)

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 1 \\ 2 & 4 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

There is no such linear combination because the system $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{y}$ has no solutions.

(c) There are a number of ways to show this. We do it directly—let $\mathbf{v} = [x, y, z]$ be an arbitrary vector, and set up a system expressing $\mathbf{v}$ as a linear combination of $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & x \\ 1 & -1 & 1 & y \\ 1 & 1 & -1 & z \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{y+z}{2} \\ 0 & 1 & 0 & \frac{x+z}{2} \\ 0 & 0 & 1 & \frac{x+y}{2} \end{array}\right).$$

So

$$\mathbf{v} = \frac{y+z}{2}\mathbf{a} + \frac{x+z}{2}\mathbf{b} + \frac{x+y}{2}\mathbf{c}.$$

OTHER METHODS:

- Show the matrix of these columns is invertible (and explain why this will always solve the system corresponding to expressing a matrix as a combination of $\mathbf{a}, \mathbf{b}, \mathbf{c}$
- Show that the standard basis vectors are each linear combinations of $\mathbf{a}, \mathbf{b}, \mathbf{c}$
- etc.

NOTE: I am using "$\rightsquigarrow$" above only to abbreviate a SERIES of row operations. It is not an official symbol, and you should SHOW your steps. If you prefer, it is shorthand for "$\equiv \cdots \equiv$"

2. (a) A 4×4 matrix A has determinant -3.
What is the determinant of adj A?

(b) Find the adjoint of matrix $B = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

(c) Find $|B|$; use this and your answer from 2b to find B^{-1}.

(d) Use your answer from 2c to solve the system

$$\begin{array}{rcrcrcr} x_1 & & & + & 2x_3 & = & 0 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 3 \\ & & x_2 & + & x_3 & = & -3 \end{array}$$

Sol:

(a) We know that $A(\text{adj } A) = (\det A)I_4$. Taking determinants on both sides gives $(\det A)(\det \text{adj } A) = (\det A)^4$. Therefore, $\det \text{adj } A = (\det A)^3 = (-3)^3 = -27$.

(b) $\text{adj } (B) = \begin{pmatrix} 3-1 & -(2-0) & 2-0 \\ -(0-2) & 1-0 & -(1-0) \\ 0-6 & -(1-4) & 3-0 \end{pmatrix}^{\top} = \begin{pmatrix} 2 & 2 & -6 \\ -2 & 1 & 3 \\ 2 & -1 & 3 \end{pmatrix}$

(c) $|B| = 6$ (for example, in checking that the answer is right in 2b by multiplying it by B one obtains $6I$); Thus, $B^{-1} = \frac{1}{|B|}\text{adj } B = \frac{1}{6}\begin{pmatrix} 2 & 2 & -6 \\ -2 & 1 & 3 \\ 2 & -1 & 3 \end{pmatrix}$.

(d) The system, in matrix form, is $BX = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}$, so the solution is

$$X = B^{-1}\begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 2 & 2 & -6 \\ -2 & 1 & 3 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix}.$$

3. Use the *Matrix Inversion Algorithm* (also called the *Direct Method* in the text) to find M^{-1}, where $M = \begin{pmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 3 & 2 & 2 \end{pmatrix}$.

Sol: $(M|I) = \left(\begin{array}{ccc|ccc} 4 & 2 & 3 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 3 & 2 & 2 & 0 & 0 & 1 \end{array}\right)$

$$\equiv \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & -1 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 3 & 2 & 2 & 0 & 0 & 1 \end{array}\right) \equiv \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & -1 & 0 \\ 0 & 6 & -1 & -3 & 4 & 0 \\ 0 & 5 & -1 & -3 & 3 & 1 \end{array}\right)$$

$$\equiv \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 5 & -1 & -3 & 3 & 1 \end{array}\right) \equiv \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -3 & -2 & 6 \end{array}\right)$$

$$\equiv \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -2 & 5 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -3 & -2 & 6 \end{array}\right) \equiv \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -2 & 5 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 & 2 & -6 \end{array}\right)$$

So $M^{-1} = \begin{pmatrix} -2 & -2 & 5 \\ 0 & 1 & -1 \\ 3 & 2 & -6 \end{pmatrix}$.

NOTES: You should still be in the habit of annotating but in an MIA calculation as long as you show all steps we *might* let it go. We do expect that you stick with our instructions for avoiding circular operations and only combine row operations that we have allowed, in a single "step".

Your answer should NOT be the final 6×3 array. Also, don't just circle the last part etc. etc. Write your answer on a separate line that *declares* exactly what you are claiming M^{-1} to be.

This is a question *nobody* should get wrong because you can always quickly and easily check to see whether your answer is correct.

4. Show that one of the following sets of vectors is dependent, and that the other one is independent.

$$S_1 = \{[1,1,0,0],[1,0,1,0],[0,1,0,1],[0,0,1,1]\}$$

$$S_2 = \{[0,1,1,1],[1,0,1,1],[1,1,0,1],[1,1,1,0]\}$$

Sol: Since these are n n-tuples ($n = 4$) this can be determined by $n \times n$ determinants. (In general the question is reduced to whether or not the matrix with the vectors as columns has rank equal to the number of columns – that is a leading one in each column, in RREF). The corresponding determinants are

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

and

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & -1 & -2 \end{vmatrix}$$

$= (-2)^2 - (-1)^2 = 3 \neq 0$, respectively.

Therefore, S_1 is linearly dependent, while S_2 is linearly independent.

5. Let $\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}$, be vectors and suppose each of $\mathbf{x}, \mathbf{y}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

   Prove that, if $\mathbf{w}$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$, then $\mathbf{w}$ is also a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Sol: Unmarked.

6. (a) $S : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and $\mathbf{x}$ are vectors such that

$$S(\mathbf{u}) = [1, 3, -2],\ S(\mathbf{v}) = [2, 2, 0], \text{ and } S(\mathbf{w}) = [-1, 1, 1].$$

If $\mathbf{x} = \mathbf{u} + 2\mathbf{v} + 3\mathbf{w}$, find $S(\mathbf{x})$.

(b) Suppose $T : \mathbb{R}^4 \to \mathbb{R}^4$ is a linear transformation such that

$$T[1, 1, 1, 1] = [1, 2, 3, 4],\ T[0, 1, 1, 1] = [2, 2, 0, 2],$$

$$T[0, 0, 1, 1] = [2, -1, 3, 2], \text{ and } T[0, 0, 0, 1] = [0, 0, 0, 0].$$

i. Find the matrix of T.

ii. Use the matrix of T to find $T[3, 2, 1, 1]$.

Sol:

(a)

$$\begin{aligned} S(\mathbf{x}) &= S(\mathbf{u} + 2\mathbf{v} + 3\mathbf{w}) = S(\mathbf{u}) + 2S(\mathbf{v}) + 3S(\mathbf{w}) \\ &= [1, 3, -2] + 2[2, 2, 0] + 3[-1, 1, 1] = [2, 10, 1] \end{aligned}$$

(b) (i) We calculate that

$$T[1, 0, 0, 0] = T[1, 1, 1, 1] - T[0, 1, 1, 1] = [1, 2, 3, 4] - [2, 2, 0, 2] = [-1, 0, 3, 2]$$

and

$$T[0, 1, 0, 0] = T[0, 1, 1, 1] - T[0, 0, 1, 1] = [2, 2, 0, 2] - [2, -1, 3, 2] = [0, 3, -3, 0]$$

and

$$T[0, 0, 1, 0] = T[0, 0, 1, 1] - T[0, 0, 0, 1] = [2, -1, 3, 2] - [0, 0, 0, 0] = [2, -1, 3, 2],$$

while $T[0, 0, 0, 1] = [0, 0, 0, 0]$ is given. Writing columns as vectors we thus obtain the matrix, A, of T as follows.

$$A = \left(T\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \middle| T\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \middle| T\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \middle| T\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right) = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & 3 & -1 & 0 \\ 3 & -3 & 3 & 0 \\ 2 & 0 & 2 & 0 \end{pmatrix}.$$

(ii) We calculate that

$$T\begin{pmatrix}3\\2\\1\\1\end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & 3 & -1 & 0 \\ 3 & -3 & 3 & 0 \\ 2 & 0 & 2 & 0 \end{pmatrix}\begin{pmatrix}3\\2\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\5\\6\\8\end{pmatrix}.$$