MATH 1210 Assignment 1 Winter 2013 Due date: January 25

1. Use mathematical induction on positive integer n to prove each of the following:

(a) $1(3)+2(3^2)+3(3^3)+\cdots+n(3^n)=\frac{1}{4}\,[\,(2n-1)3^{n+1}+3\,]$, **for** $n\geq 1$**;**

(b) $1^2+2^2+3^2+\cdots+(3n)^2=\frac{1}{2}\,[n(3n+1)(6n+1)]$, **for** $n\geq 1$**;**

(c) $5^{2n-1}+1$ **is divisible by** 6 **for** $n\geq 1$**.**

Solution:

(a) If $n=1$ **then** $1(3)=3$ **and** $\frac{1}{4}[(2-1)3^2+3]=\frac{1}{4}(12)=3$, **so it is valid for** $n=1$. **We assume that it is valid for** $n=k\geq 1$ **that is**

$$1(3)+2(3^2)+\cdots+k(3^k)=\frac{1}{4}[\,(2k-1)3^{k+1}+3]\qquad(*)$$

We need to prove that it is valid for $n=k+1$ **that is we must verify that**

$$1(3)+2(3^2)+\cdots+(k+1)(3^{k+1})=\frac{1}{4}[\,(2k+1)3^{k+2}+3]\,.$$

But

$$\begin{aligned}1(3)+2(3^2)+\cdots+(k+1)(3^{k+1})&=[1(3)+2(3^2)+\cdots+k(3^k)]+(k+1)(3^{k+1})\\&=\frac{1}{4}[\,(2k-1)3^{k+1}+3]+(k+1)(3^{k+1})\quad\textbf{by }(*)\\&=\frac{1}{4}[\,(2k-1)3^{k+1}+3+4(k+1)(3^{k+1})]\\&=\frac{1}{4}[\,3^{k+1}\big((2k-1)+4(k+1)\big)+3]\\&=\frac{1}{4}[\,3^{k+1}(6k+3)+3]\\&=\frac{1}{4}[\,(2k+1)3^{k+2}+3]\,.\end{aligned}$$

Therefore by the principle of mathematical induction the formula is valid for all $n\geq 1$**.**

(b) One can solve it in two different ways:
Solution (1): If $n=1$ **then** $1^2+2^2+3^2=14$ **and** $\frac{1}{2}[1(3+1)(6+1)]=\frac{28}{2}=14$, **so it is valid for** $n=1$. **We assume that it is valid for** $n=k\geq 1$ **that is**

$$1^2+2^2+3^2+\cdots+(3k)^2=\frac{1}{2}[\,k(3k+1)(6k+1)]\qquad(*)$$

We need to prove that it is valid for $n=k+1$ **that is we must verify that**

$$\begin{aligned}1^2+2^2+3^2+\cdots+\big(3(k+1)\big)^2&=\frac{1}{2}[\,(k+1)\big(3(k+1)+1\big)\big(6(k+1)+1\big)]\\&=\frac{1}{2}[\,(k+1)(3k+4)(6k+7)]\\&=\frac{1}{2}(18k^3+27k^2+10k+1)\,.\end{aligned}$$

But

$$\begin{aligned}
&1^2+2^2+3^2+\cdots+\big(3(k+1)\big)^2\\
&=[1^2+2^2+3^2+\cdots+(3k)^2]+(3k+1)^2+(3k+2)^2+(3k+3)^2\\
&=\frac{1}{2}[\,k(3k+1)(6k+1)]+(27k^2+36k^2+14)\quad \textbf{by } (*)\\
&=\frac{1}{2}(18k^3+27k^2+10k+1)+(27k^2+36k^2+14)\\
&=\frac{1}{2}[\,(18k^3+27k^2+10k+1)+2(27k^2+36k^2+14)]\\
&=\frac{1}{2}(18k^3+27k^2+10k+1)\,.
\end{aligned}$$

Therefore by the principle of mathematical induction the formula is valid for all $n\geq 1$.
Solution (2): Let $m=3n$ then it becomes

$$1^2+2^2+3^2+\cdots+(m)^2=\frac{1}{2}\Big[\,\frac{m}{3}\big(3(\frac{m}{3})+1\big)\big(6(\frac{m}{3})+1\big)\Big]=\frac{1}{6}[m(m+1)(2m+1)]$$

Now by an easy induction on $m\geq 3$ one can prove that

$$1^2+2^2+3^2+\cdots+(m)^2=\frac{1}{6}[m(m+1)(2m+1)]$$

(You do it! and note that it is for all integers bigger than or equal to 3 not just for multiples of 3). Then conclude that since it is valid for all $m\geq 3$ in particular it is valid for $m=3n$ which means $1^2+2^2+3^2+\cdots+(3n)^2=\dfrac{1}{2}[\,n(3n+1)(6n+1)]$.

(c) **If $n=1$ then $5^{2-1}+1=6$ which is divisible by 6, so it is valid for $n=1$. We assume that it is valid for $n=k\geq 1$ that is $5^{2k-1}+1$ is divisible by 6. We need to prove that it is valid for $n=k+1$ that is we must verify that $5^{2k+1}+1$ is divisible by 6. But**

$$5^{2k+1}+1=5^{2k+1}+5^2-5^2+1=5^2(5^{2k-1}+1)-24\qquad(*)$$

Sine 24 is divisible by 6 and also by induction hypothesis $5^{2k-1}+1$ is divisible by 6 so the right hand side of $(*)$ is divisible by 6 and therefore the left hand side is divisible by 6.
Therefore by the principle of mathematical induction it is valid for all $n\geq 1$.

2. Let a be a real number, use mathematical induction on positive integer $n\geq 1$ to prove that

$$a^{n+1}-1=(a-1)(a^n+a^{n-1}+\cdots+a+1)$$

Do not use any other method.

Solution: If $n=1$ then $a^{1+1}-1=a^2-1$ and $(a-1)(a+1)=a^2-1$, so it is valid for $n=1$. We assume that it is valid for $n=k\geq 1$ that is

$$a^{k+1}-1=(a-1)(a^k+a^{k-1}+\cdots+a+1)\qquad(*)\,.$$

We need to prove that it is valid for $n=k+1$ that is we must verify that

$$a^{k+2}-1=(a-1)(a^{k+1}+a^k+\cdots+a+1)\,.$$

But

$$\begin{aligned} a^{k+2}-1 = a^{k+2}-a^{k+1}+a^{k+1}-1 &= a^{k+1}(a-1)+(a^{k+1}-1) \\ &= a^{k+1}(a-1)+(a-1)(a^{k}+a^{k-1}+\cdots+a+1) \quad \textbf{by } (*) \\ &= (a-1)(a^{k+1}+a^{k}+a^{k-1}+\cdots+a+1)\,. \end{aligned}$$

Therefore by the principle of mathematical induction the formula is valid for all $n \geq 1$.

3. First write sigma form of $(13)^2+(25)^2+(37)^2+\cdots+(24n+1)^2$ and then use identities $\sum_{k=1}^{m} k = \frac{1}{2}\left[m(m+1)\right]$ and $\sum_{k=1}^{m} k^2 = \frac{1}{6}\left[m(m+1)(2m+1)\right]$ to prove that

$$(13)^2+(25)^2+(37)^2+\cdots+(24n+1)^2 = 2n\left(192n^2+168n+37\right).$$

Solution: Since $13 = 12+1$ **,** $25 = 12(2)+1$ **and** $(24n+1) = [12(2n)+1]$ **so** $(13)^2+(25)^2+(37)^2+\cdots+(24n+1)^2 = \sum_{j=1}^{2n}(12j+1)^2$**. Now using given formulas we have**

$$\begin{aligned} &\sum_{j=1}^{2n}(12j+1)^2 = \sum_{j=1}^{2n}(144j^2+24j+1) \\ &= 144\sum_{j=1}^{2n} j^2 + 24\sum_{j=1}^{2n} j + \sum_{j=1}^{2n} 1 \\ &= 144[\tfrac{1}{6}(2n)(2n+1)(4n+1)] + 24[\tfrac{1}{2}(2n)(2n+1)] + 2n \\ &= 2n[24(2n+1)(4n+1)+12(2n+1)+1] \\ &= 2n(192n^2+168n+37)\,. \end{aligned}$$

4. Express the sum in sigma notation with summation index starting from 1.

(a) $2+\frac{3}{\sqrt{2}}+\frac{4}{\sqrt{3}}+\frac{5}{2}+\cdots+\frac{101}{10}$ **;**

(b) $2+\frac{1}{\sqrt{2}}+\frac{2}{3\sqrt{3}}+\frac{1}{4}+\frac{2}{5\sqrt{5}}+\cdots+\frac{1}{10\sqrt{20}}$ **;**

(c) $1-\frac{2}{9}+\frac{6}{25}-\frac{24}{49}+\frac{120}{81}-\cdots$ **.**

Solution:

(a) $2+\frac{3}{\sqrt{2}}+\frac{4}{\sqrt{3}}+\frac{5}{2}+\cdots+\frac{101}{10} = \sum_{n=1}^{100}\frac{n+1}{\sqrt{n}}$

(b) $2+\frac{1}{\sqrt{2}}+\frac{2}{3\sqrt{3}}+\frac{1}{4}+\frac{2}{5\sqrt{5}}+\cdots+\frac{1}{10\sqrt{20}} = \sum_{n=1}^{20}\frac{2}{n\sqrt{n}}$

(c) $1-\frac{2}{9}+\frac{6}{25}-\frac{24}{49}+\frac{120}{81}-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,n!}{(2n-1)^2}$

5. Use formulas $\sum_{k=1}^{m} k=\frac{1}{2}\,[m(m+1)]$ **,** $\sum_{k=1}^{m} k^2=\frac{1}{6}\,[m(m+1)(2m+1)]$ **and** $\sum_{k=1}^{m} k^3=\frac{1}{4}\,[m^2(m+1)^2]$ **to evaluate each of the following sum :**

(a) $\sum_{\ell=31}^{41}\,[\frac{1}{22}(\ell-32)^2-\frac{2}{11}]$ **;**

(b) $\sum_{j=-11}^{88}\,\big[(j+12)^3+(j+12)^2-j-12\big]$ **.**

Solution:

(a)

$$\begin{aligned}
\sum_{\ell=31}^{41}[\frac{1}{22}(\ell-32)^2-\frac{2}{11}] &= \frac{1}{22}\sum_{\ell=31}^{41}[(\ell-32)^2-4]\\
&= \frac{1}{22}\sum_{\ell=1}^{11}[(\ell+30-32)^2-4]=\frac{1}{22}\sum_{\ell=1}^{11}[(\ell-2)^2-4]\\
&= \frac{1}{22}\sum_{\ell=1}^{11}[\ell^2-4\ell+4-4]=\frac{1}{22}\Big[\sum_{\ell=1}^{11}\ell^2-4\sum_{\ell=1}^{11}\ell\Big]\\
&= \frac{1}{22}\Big[\frac{1}{6}(11)(11+1)(22+1)-4\,(\frac{1}{2})(11)(11+1)\Big]\\
&= \frac{1}{22}\Big[\frac{1}{6}(11)(12)(23)-4\,(\frac{3}{6})(11)(12)\Big]\\
&= \frac{1}{22}\Big[\frac{1}{6}(11)(12)\,(23-12)\Big]\\
&= \frac{1}{22}\Big[(22)\,(11)\Big]\\
&= 11\,.
\end{aligned}$$

(b)

$$\begin{aligned}
&\sum_{j=-11}^{88}\,[(j+12)^3+(j+12)^2-j-12]\\
&= \sum_{j=1}^{100}[(j-12+12)^3+(j-12+12)^2-(j-12)-12]\\
&= \sum_{j=1}^{100}[j^3+j^2-j]=\sum_{j=1}^{100}j^3+\sum_{j=1}^{100}j^2-\sum_{j=1}^{100}j\,.
\end{aligned}$$

Now using given formulas we have

$$
\begin{aligned}
&\sum_{j=1}^{100} j^3 + \sum_{j=1}^{100} j^2 - \sum_{j=1}^{100} j \\
&= \frac{1}{4}(100)^2(100+1)^2 + \frac{1}{6}(100)(100+1)(200+1) - \frac{1}{2}(100)(100+1) \\
&= \frac{1}{12}(100)(101)\Big[\,3(100)(101) + 2(201) - 6\Big] \\
&= 25835800\,.
\end{aligned}
$$

6. Prove that $\displaystyle\sum_{\ell=n}^{\ell=2n}(\ell+1) = \frac{1}{2}\,[(n+1)(3n+2)]$ **with each of the following two methods:**

(a) **By mathematical induction on positive integer** $n \geq 1$ **;**

(b) **By using the identity** $\displaystyle\sum_{k=1}^{m} k = \frac{1}{2}\,[m(m+1)]$**.**

Solution:

(a) If $n = 1$ **then** $\displaystyle\sum_{\ell=1}^{\ell=2}(\ell+1) = (1+1)+(2+1) = 5$ **and** $\frac{1}{2}[(1+1)(3+2)] = 5$**, so it is valid for** $n=1$**. We assume that it is valid for** $n = k \geq 1$ **that is**

$$\sum_{\ell=k}^{\ell=2k}(\ell+1) = \frac{1}{2}[(k+1)(3k+2)] \qquad (*)$$

We need to prove that it is valid for $n = k+1$ **that is we must verify that**

$$\sum_{\ell=k+1}^{\ell=2(k+1)}(\ell+1) = \frac{1}{2}[(k+2)(3k+5)]\,.$$

But

$$
\begin{aligned}
\sum_{\ell=k+1}^{\ell=2(k+1)}(\ell+1) &= -(k+1) + \sum_{\ell=k}^{\ell=2(k+1)}(\ell+1) \\
&= -(k+1) + \sum_{\ell=k}^{\ell=2k}(\ell+1) + \big((2k+1)+1\big) + \big((2k+2)+1\big) \\
&= \sum_{\ell=k}^{\ell=2k}(\ell+1) + (3k+4) \\
&= \frac{1}{2}[(k+1)(3k+2)] + (3k+4) \qquad \textbf{by } (*) \\
&= \frac{1}{2}[(k+1)(3k+2) + 2(3k+4)] \\
&= \frac{1}{2}(3k^2 + 11k + 10) \\
&= \frac{1}{2}[(k+2)(3k+5)]\,.
\end{aligned}
$$

Therefore by the principle of mathematical induction the formula is valid for all $n \geq 1$.

(b) First we note that

$$\sum_{\ell=n}^{\ell=2n}(\ell+1) = \sum_{\ell=n}^{\ell=2n}\ell + \sum_{\ell=n}^{\ell=2n}1 = \sum_{\ell=n}^{\ell=2n}\ell + (2n-n+1) = \sum_{\ell=n}^{\ell=2n}\ell + (n+1) \qquad (*)$$

Then using given formula we have

$$\begin{aligned}\sum_{\ell=n}^{\ell=2n}\ell = \sum_{\ell=1}^{\ell=2n}\ell - \sum_{\ell=1}^{\ell=n-1}\ell &= \frac{1}{2}[2n(2n+1)] - \frac{1}{2}[(n-1)(n-1+1)] \\ &= \frac{1}{2}[2n(2n+1) - n(n-1)] \\ &= \frac{1}{2}[3n^2+3n]\end{aligned}$$

By substitution in $(*)$ we get

$$\begin{aligned}\sum_{\ell=n}^{\ell=2n}(\ell+1) = \sum_{\ell=n}^{\ell=2n}\ell + (n+1) &= \frac{1}{2}[3n^2+3n] + (n+1) \\ &= \frac{1}{2}[3n^2+3n+2(n+1)] \\ &= \frac{1}{2}(3n^2+5n+2) \\ &= \frac{1}{2}[(n+1)(3n+2)]\,.\end{aligned}$$

7. For each of the sums $\displaystyle\sum_{n=4}^{203}\frac{9^{n-3}+5^{n-1}}{(n-3)!}$ and $\displaystyle\sum_{n=0}^{199}\frac{3^{2n+2}+5^{n+1}}{(n+1)!}$, change the summation index to 1 and then use it to write $\displaystyle\sum_{n=4}^{203}\frac{9^{n-3}+5^{n-1}}{(n-3)!} - \sum_{n=0}^{199}\frac{3^{2n+2}+5^{n+1}}{(n+1)!}$ as one sum. Simplify as much as possible.

Solution:

$$\sum_{n=4}^{203}\frac{9^{n-3}+5^{n-1}}{(n-3)!} = \sum_{n=1}^{200}\frac{9^{n+3-3}+5^{n+3-1}}{(n+3-3)!} = \sum_{n=1}^{200}\frac{9^n+5^{n+2}}{n!}$$

$$\text{and } \sum_{n=0}^{199}\frac{3^{2n+2}+5^{n+1}}{(n+1)!} = \sum_{n=1}^{200}\frac{3^{2(n-1)+2}+5^{n-1+1}}{(n-1+1)!} = \sum_{n=1}^{200}\frac{3^{2n}+5^n}{n!}\,.$$

In order to simplify the given expression we have

$$\sum_{n=4}^{203}\frac{9^{n-3}+5^{n-1}}{(n-3)!} - \sum_{n=0}^{199}\frac{3^{2n+2}+5^{n+1}}{(n+1)!}$$

$$= \sum_{n=1}^{200}\frac{9^n+5^{n+2}}{n!} - \sum_{n=1}^{200}\frac{3^{2n}+5^n}{n!} = \sum_{n=1}^{200}\frac{9^n+5^{n+2}-3^{2n}-5^n}{n!} = \sum_{n=1}^{200}\frac{9^n-9^n+5^n(5^2-1)}{n!} = 24\sum_{n=1}^{200}\frac{5^n}{n!}\,.$$