MATH 1210 Winter 2013 Assignment 3 Solutions

1. We start by showing it's true for $n = 1$. The left hand side is

$$A = \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix}.$$

The right hand side is

$$\frac{1}{7}\begin{bmatrix} 6(-1)^1 + 6^1 & -2(-1)^1 + 2 \cdot 6^1 \\ -3(-1)^1 + 3 \cdot 6^1 & (-1)^1 + 6 \cdot 6^1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 0 & 14 \\ 21 & 35 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix}.$$

Since the left and right hand sides match, the formula is true for $n = 1$.

Suppose the formula is true for an integer $n = k$. Hence

$$A^k = \frac{1}{7}\begin{bmatrix} 6(-1)^k + 6^k & -2(-1)^k + 2 \cdot 6^k \\ -3(-1)^k + 3 \cdot 6^k & (-1)^k + 6 \cdot 6^k \end{bmatrix}$$

We need to show the formula is true for $n = k + 1$. That is

$$A^{k+1} = \frac{1}{7}\begin{bmatrix} 6(-1)^{k+1} + 6^{k+1} & -2(-1)^{k+1} + 2 \cdot 6^{k+1} \\ -3(-1)^{k+1} + 3 \cdot 6^{k+1} & (-1)^{k+1} + 6 \cdot 6^{k+1} \end{bmatrix}$$

Thus

$$\begin{aligned}
LHS &= A^{k+1} \\
&= A(A^k) \\
&= \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix}\left(\frac{1}{7}\begin{bmatrix} 6(-1)^{k+1} + 6^{k+1} & -2(-1)^{k+1} + 2 \cdot 6^{k+1} \\ -3(-1)^{k+1} + 3 \cdot 6^{k+1} & (-1)^{k+1} + 6 \cdot 6^{k+1} \end{bmatrix}\right) \\
&= \frac{1}{7}\begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 6(-1)^{k+1} + 6^{k+1} & -2(-1)^{k+1} + 2 \cdot 6^{k+1} \\ -3(-1)^{k+1} + 3 \cdot 6^{k+1} & (-1)^{k+1} + 6 \cdot 6^{k+1} \end{bmatrix} \\
&= \frac{1}{7}\begin{bmatrix} 0 + 2\big(-3(-1)^k + 3 \cdot 6^k\big) & 0 + 2\big((-1)^k + 6 \cdot 6^k\big) \\ 3\big(6(-1)^k + 6^k\big) + 5\big(-3(-1)^k + 3 \cdot 6^k\big) & 3\big(-2(-1)^k + 2 \cdot 6^k\big) + 5\big((-1)^k + 6 \cdot 6^k\big) \end{bmatrix} \\
&= \frac{1}{7}\begin{bmatrix} -6(-1)^k + 6 \cdot 6^k & 2(-1)^k + 12 \cdot 6^k \\ 18(-1)^k + 3 \cdot 6^k - 15(-1)^k + 15 \cdot 6^k & -6(-1)^k + 6 \cdot 6^k + 5 \cdot (-1)^k + 30 \cdot 6^k \end{bmatrix} \\
&= \frac{1}{7}\begin{bmatrix} -6(-1)^k + 6 \cdot 6^k & 2(-1)^k + 12 \cdot 6^k \\ 3(-1)^k + 18 \cdot 6^k & -1(-1)^k + 36 \cdot 6^k \end{bmatrix} \\
&= \frac{1}{7}\begin{bmatrix} 6(-1)^{k+1} + 6^{k+1} & -2(-1)^{k+1} + 2 \cdot 6^{k+1} \\ -3(-1)^{k+1} + 3 \cdot 6^{k+1} & (-1)^{k+1} + 6 \cdot 6^{k+1} \end{bmatrix} \\
&= RHS.
\end{aligned}$$

Hence the formula is true for $n = k + 1$.

By the principle of mathematical induction, the formula is true for $n \geq 1$.

2. (a) Let's start by finding $\mathbf{u} \times \mathbf{v}$.

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \langle 2,3,4\rangle \times \langle 1,2,3\rangle \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix} \\
&= \begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \hat{\mathbf{k}} \\
&= (9-8)\hat{\mathbf{i}} - (6-4)\hat{\mathbf{j}} + (4-3)\hat{\mathbf{k}} \\
&= \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 1\hat{\mathbf{k}} \\
&= \langle 1,-2,1\rangle
\end{aligned}$$

Now, let's find $\hat{\mathbf{u}}$.

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\langle 2,3,4\rangle}{\sqrt{2^2+3^2+4^2}} = \frac{\langle 2,3,4\rangle}{\sqrt{29}}$$

Hence

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} - 2\hat{\mathbf{u}} &= \langle 1,-2,1\rangle - 2\left(\frac{\langle 2,3,4\rangle}{\sqrt{29}}\right) \\
&= \left\langle 1 - \frac{4}{\sqrt{29}}, -2 - \frac{6}{\sqrt{29}}, 1 - \frac{8}{\sqrt{29}} \right\rangle \text{ or} \\
&= \left\langle \frac{\sqrt{29}-4}{\sqrt{29}}, \frac{-2\sqrt{29}-6}{\sqrt{29}}, \frac{\sqrt{29}-8}{\sqrt{29}} \right\rangle.
\end{aligned}$$

(b)

$$2\mathbf{u} - 3\mathbf{v} = 2\langle 2,3,4\rangle - 3\langle 1,2,3\rangle = \langle 1,0,-1\rangle$$

and

$$3\mathbf{u} - 2\mathbf{v} = 3\langle 2,3,4\rangle - 2\langle 1,2,3\rangle = \langle 4,5,6\rangle$$

Hence if θ is the angle between $2\mathbf{u} - 3\mathbf{v}$ and $3\mathbf{u} - 2\mathbf{v}$, we get that

$$\begin{aligned}
\cos\theta &= \frac{\langle 1,0,-1\rangle \cdot \langle 4,5,6\rangle}{|\langle 1,0,-1\rangle|\,|\langle 4,5,6\rangle|} \\
&= \frac{1(4)+0(5)-1(6)}{\sqrt{1^2+0^2+(-1)^2}\sqrt{4^2+5^2+6^2}} \\
&= -\frac{2}{\sqrt{2}\sqrt{77}} \\
&= -\frac{2}{\sqrt{154}}
\end{aligned}$$

3. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in 3 dimensions.

   (a) Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$. Then

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{u} &= \langle u_1, u_2, u_3 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\
&= u_1^2 + u_2^2 + u_3^2 \\
&= \left(\sqrt{u_1^2 + u_2^2 + u_3^2}\right)^2 \\
&= |\mathbf{u}|^2
\end{aligned}$$

   (b) Using part (a) we get that

$$\begin{aligned}
|\mathbf{u}+\mathbf{v}|^2 + |\mathbf{u}-\mathbf{v}|^2 &= (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v}) \\
&= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} \\
&= 2\mathbf{u}\cdot\mathbf{u} + 2\mathbf{v}\cdot\mathbf{v} \\
&= 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2
\end{aligned}$$

4. (a) There are two options for this question. Solution 1 is using techniques from Section 5.2.

   l_1 is on both planes, and therefore it's perpendicular to both normals. The normal to the first plane is $\mathbf{n_1} = \langle 1, -2, 2 \rangle$ and the normal to the second plane is $\mathbf{n_2} = \langle 3, 1, 1 \rangle$. Hence the vector parallel to l_1 is

$$\begin{aligned}
\mathbf{n_1} \times \mathbf{n_2} &= \langle 1, -2, 2 \rangle \times \langle 3, 1, 1 \rangle \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -2 & 2 \\ 3 & 1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \hat{\mathbf{k}} \\
&= (-2-2)\hat{\mathbf{i}} - (1-6)\hat{\mathbf{j}} + (1-(-6))\hat{\mathbf{k}} \\
&= -4\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 7\hat{\mathbf{k}} \\
&= \langle -4, 5, 7 \rangle.
\end{aligned}$$

   Now all we need is a point on the line. For example, let $z = 0$. Then $x - 2y = 5$ and $3x + y = 3$. Taking the first equation plus 2 times the second yields $7x = 11$. Hence $x = 11/7, y = -12/7$. So a point on the line is

$$\left(\frac{11}{7}, -\frac{12}{7}, 0\right)$$

   Therefore parametric equations are

$$x = -4t + \frac{11}{7}, \quad y = 5t - \frac{12}{7}, \quad z = 7t.$$

Option 2 is by using systems of linear equations.

$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 3 & 1 & 1 & 3 \end{array}\right]$ Using $R_2 \to R_2 - 3R_1$ yields

$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 7 & -5 & -12 \end{array}\right]$ Using $R_2 \to \frac{1}{7}R_2$ yields

$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 1 & -5/7 & -12/7 \end{array}\right]$ Using $R_1 \to R_1 + 2R_2$ yields

$\left[\begin{array}{ccc|c} 1 & 0 & 4/7 & 11/7 \\ 0 & 1 & -5/7 & -12/7 \end{array}\right]$ Let $z = t$, then

$$x = \frac{11}{7} - \frac{4}{7}t, \quad -\frac{12}{7} + \frac{5}{7}t, \quad z = t.$$

(b) Looking at the first two parts of the symmetric equations of l_2, we insert the values of x and y.

$$\begin{aligned} \frac{-4t + \frac{11}{7} - 1}{3} &= \frac{2\left(5t - \frac{12}{7}\right) + 2}{4} \\ \frac{-4t + \frac{4}{7}}{3} &= \frac{10t - \frac{10}{7}}{4} \\ -16t + \frac{16}{7} &= 30t - \frac{30}{7} \\ \frac{46}{7} &= 46t \\ t &= \frac{1}{7}. \end{aligned}$$

Plugging this value of t into l_1 yields

$$x = 1, y = -1, z = 1$$

which satisfies the symmetric equations. Hence the point of intersection is $(1, -1, 1)$.

(c) Let

$$s = \frac{x-1}{3} = \frac{2y+2}{4} = \frac{1-z}{2}.$$

Then we get

$$x = 3s + 1, \quad y = 2s - 1 \quad z = -2s + 1$$

Plugging this into the plane yields

$$2(3s+1)+(2s-1)+(-2s+1)=14 \Rightarrow 6s+2=14 \Rightarrow s=2.$$

Hence plugging this back into

$$x=3(2)+1=7, \quad y=2(2)-1=3, \quad z=-2(2)+1=-3.$$

Hence the point of intersection is $(7,3,-3)$.

(d) The plane exists because we know from (b) that the lines intersect. The vector parallel to l_1 is $\langle -4,5,7\rangle$. The vector parallel to line l_2 is $\langle 3,2,-2\rangle$. Hence the normal vector to the plane is

$$\begin{aligned}
\langle -4,5,7\rangle \times \langle 3,2,-2\rangle &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -4 & 5 & 7 \\ 3 & 2 & -2 \end{vmatrix} \\
&= \begin{vmatrix} 5 & 7 \\ 2 & -2 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} -4 & 7 \\ 3 & -2 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} -4 & 5 \\ 3 & 2 \end{vmatrix} \hat{\mathbf{k}} \\
&= (-10-14)\hat{\mathbf{i}} - (8-21)\hat{\mathbf{j}} + (-8-15)\hat{\mathbf{k}} \\
&= -24\hat{\mathbf{i}} + 13\hat{\mathbf{j}} - 23\hat{\mathbf{k}}
\end{aligned}$$

We can take any point on either line (for example $(1,-1,1)$.) Hence the equation of the plane is

$$-24(x-1)+13(y+1)-23(z-1)=0.$$

5. (a) There are many ways to get into RREF. Here is one

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 6 \\ 1 & -1 & 3 & -4 \\ 3 & 7 & -5 & 16 \end{array}\right] \text{ Using } R_1 \leftrightarrow R_2 \text{ yields}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -4 \\ 2 & 3 & -1 & 6 \\ 3 & 7 & -5 & 16 \end{array}\right] \text{ Using } R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - 2R_1 \text{ yields}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -4 \\ 0 & 5 & -7 & 14 \\ 0 & 10 & -14 & 28 \end{array}\right] \text{ Using } R_3 \to R_3 - 2R_2 \text{ yields}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -4 \\ 0 & 5 & -7 & 14 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ Using } R_2 \to \frac{1}{5}R_2 \text{ yields}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -4 \\ 0 & 1 & -7/5 & 14/5 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ Using } R_2 \to \frac{1}{5}R_2 \text{ yields}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -4 \\ 0 & 1 & -7/5 & 14/5 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ Using } R_1 \to R_1 + R_2 \text{ yields}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 8/5 & -6/5 \\ 0 & 1 & -7/5 & 14/5 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ This leads to the solutions}$$

$$x = -\frac{6}{5} - \frac{8}{5}z \quad y = \frac{14}{5} + \frac{7}{5}z$$

where z is arbitrary.

(b) There are many ways to get into RREF. Here is one

$$\left[\begin{array}{ccc|c} 3 & 1 & -3 & 6 \\ 1 & 2 & 3 & 4 \\ 3 & -4 & -15 & 2 \end{array}\right] \text{ Using } R_1 \leftrightarrow R_2 \text{ yields}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 3 & 1 & -3 & 6 \\ 3 & -4 & -15 & 2 \end{array}\right] \text{ Using } R_2 \to R_2 - 3R_1 \text{ and } R_3 \to R_3 - 3R_1 \text{ yields}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -5 & -12 & -6 \\ 0 & -10 & -24 & -10 \end{array}\right] \text{ Using } R_3 \to R_3 - 2R_2 \text{ yields}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -5 & -12 & -6 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

Since the last line says $0 = 2$, which can never be satisfied, there is no solution.

(c) There are many ways to get into RREF. Here is one

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 6 \\ 3 & -2 & 3 & -1 & -4 \\ 0 & 7 & -9 & 8 & 26 \end{array}\right] \text{ Using } R_1 \to 2R_1 \text{ and then } R_1 \to R_1 - R_2 \text{ yields}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & -5 & 5 & 16 \\ 3 & -2 & 3 & -1 & -4 \\ 0 & 7 & -9 & 8 & 26 \end{array}\right] \text{ Using } R_2 \to R_2 - 3R_1 \text{ yields}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & -5 & 5 & 16 \\ 0 & -14 & 18 & -16 & -52 \\ 0 & 7 & -9 & 8 & 26 \end{array}\right] \text{ Using } R_2 \to -\frac{1}{14}R_2 \text{ yields}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & -5 & 5 & 16 \\ 0 & 1 & -9/7 & 8/7 & 26/7 \\ 0 & 7 & -9 & 8 & 26 \end{array}\right] \text{ Using } R_1 \to R_1 - 4R_2 \text{ and } R_3 \to R_3 - 7R_2 \text{ yields}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1/7 & 3/7 & 8/7 \\ 0 & 1 & -9/7 & 8/7 & 26/7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This leads to the solution

$$x = \frac{8}{7} - \frac{1}{7}z - \frac{3}{7}w, \quad y = \frac{26}{7} + \frac{9}{7}z - \frac{8}{7}w$$

where z and w are arbitrary.

6. We start by putting it into an augmented matrix and row reducing into REF.

$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 6 \\ 0 & 5 & 13 & 11 \\ 0 & -5 & a^2-29 & a-15 \end{array}\right]$ Using $R_3 \to R_3 + R_2$ yields

$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 6 \\ 0 & 5 & 13 & 11 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$ Using $R_2 \to \frac{1}{5}R_2$ yields

$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 6 \\ 0 & 1 & 13/5 & 11/5 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$ (although this last step isn't really necessary to solve the question)

Now we are in a position to solve the questions

(a) There is no solution when the last line is $0 = non{-}zero$. Hence we need $a^2-16 = 0$, but $a - 4 \neq 0$. Since $a^2 - 16 = 0$ if $a = \pm 4$. The value of a such that there is no solution is $a = -4$.

(b) This happens when there is a solution (no row has the last line of $0 = non-zero$) and there is a leading one in every column, which will be when $a^2 - 16 \neq 0$. Hence there is exactly one solution for all a where $a \neq \pm 4$.

(c) This happens when there is a solution (no row has the last line of $0 = non-zero$) and there is a column without a leading one. Hence $a^2 - 16 = 0$ but we also need that the system is consistent, hence $a - 4 = 0$ as well. This happens when $a = 4$.

(An alternative for part (b) is to find values of a where the determinant is non-zero. Although this wasn't known at the time the assignment was due.)

7. Even though this is non-linear, a quick substitution of $X = \frac{1}{x}, Y = \frac{1}{y}, Z = \frac{1}{z}$ turns the system of equations into

$$\begin{aligned} X + 2Y - 4Z &= 1 \\ 2X + 3Y + 8Z &= \frac{11}{2} \\ -X + 9Y + 10Z &= 6 \end{aligned}$$

which is linear. Hence we solve like earlier questions and put it into RREF. There are many ways to get into RREF. Here is one

$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 2 & 3 & 8 & 11/2 \\ -1 & 9 & 10 & 6 \end{array}\right]$ Using $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + R_1$ yields

$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 0 & -1 & 16 & 7/2 \\ 0 & 11 & 6 & 7 \end{array}\right]$ Using $R_2 \to -R_2$ yields

$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & -7/2 \\ 0 & 11 & 6 & 7 \end{array}\right]$ Using $R_1 \to R_1 - 2R_2$ and $R_3 \to R_3 - 11R_2$ yields

$\left[\begin{array}{ccc|c} 1 & 0 & 28 & 8 \\ 0 & 1 & -16 & -7/2 \\ 0 & 0 & 182 & 91/2 \end{array}\right]$ Using $R_3 \to \dfrac{1}{182}R_3$ yields

$\left[\begin{array}{ccc|c} 1 & 0 & 28 & 8 \\ 0 & 1 & -16 & -7/2 \\ 0 & 0 & 1 & 1/4 \end{array}\right]$ Using $R_1 \to R_1 - 28R_3$ and $R_2 \to R_2 + 16R_3$ yields

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/4 \end{array}\right].$

Hence $X = 1, Y = \dfrac{1}{2}$ and $Z = \dfrac{1}{4}$. Solving back for x, y and z yields the solution

$$x = 1, \quad y = 2, \quad z = 4.$$

8. Putting the matrix into RREF yields

$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 3 & -2 & 0 \\ -2 & -4 & 1 & 0 & -5 & 0 & 0 \\ -1 & -2 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$ Using $R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 + R_1$ yields

$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$ which is in RREF. Hence solutions would be

$$x_1 = -2x_2 - 3x_5 + 2x_6, \quad x_3 = -x_5 + 4x_6, \quad x_4 = -x_5 + x_6$$

where x_2, x_5 and x_6 are arbitrary.

Hence in matrix form we have the solutions.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_5 + 2x_6 \\ x_2 \\ -x_5 + 4x_6 \\ -x_5 + x_6 \\ x_5 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 2 \\ 0 \\ 4 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore basic solutions are

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 0 \\ 4 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

9. Since U and V are solutions to $AX = \mathbf{0}$, we know that $AU = \mathbf{0}$ and $AV = \mathbf{0}$. Any linear combination of U and V is of the form

$$kU + lV$$

for any constants k and l. This is a solution since

$$A(kU + lV) = A(kU) + A(lV) = k(AU) + l(AV) = k(\mathbf{0}) + l(\mathbf{0}) = \mathbf{0}.$$