MATH 1210 Assignment 4 Winter 2013 **Due date: April 5**

1. Determine whether each of the following matrices is invertible. If yes find the inverse and if no explain why.

$$(a)\quad A = \begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix} \qquad (b)\quad B = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix} \qquad (c)\quad C = \begin{pmatrix} 3 & 1 & 2 \\ 1 & -2 & -4 \\ -5 & 3 & 6 \end{pmatrix}$$

Solution:

(a)

$$\left(\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right) R_1 \to -R_1 \quad \Rightarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 \to -2R_1 + R_2 \\ R_2 \to -4R_1 + R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right) R_2 \to -R_3 + R_2 \quad \Rightarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & -1 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right) R_2 \to -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 1 & -1 & 2 & -1 & 1 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 \to 2R_2 + R_1 \\ \\ R_3 \to -6R_2 + R_3 \end{array} \quad \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -2 & 2 \\ 0 & 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -8 & 6 & -5 \end{array}\right) R_3 \to -R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -2 & 2 \\ 0 & 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right) \begin{array}{l} R_1 \to -R_3 + R_1 \\ R_2 \to R_3 + R_1 \end{array} \quad \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 4 & -3 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right) R_3 \to -R_3$$

Therefore $A^{-1} = \begin{pmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{pmatrix}$.

(b)

$$\left(\begin{array}{ccc|ccc} -40 & 16 & 9 & 1 & 0 & 0 \\ 13 & -5 & -3 & 0 & 1 & 0 \\ 5 & -2 & -1 & 0 & 0 & 1 \end{array}\right) R_1 \to 3R_2 + R_1 \quad \Rightarrow \left(\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 3 & 0 \\ 13 & -5 & -3 & 0 & 1 & 0 \\ 5 & -2 & -1 & 0 & 0 & 1 \end{array}\right) R_1 \to -R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & -3 & 0 \\ 13 & -5 & -3 & 0 & 1 & 0 \\ 5 & -2 & -1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 \to -13R_1 + R_2 \\ R_3 \to -5R_1 + R_3 \end{array} \quad \Rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & -3 & 0 \\ 0 & 8 & -3 & 13 & 40 & 0 \\ 0 & 3 & -1 & 5 & 15 & 1 \end{array}\right) R_2 \to -3R_3 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & -3 & 0 \\ 0 & -1 & 0 & -2 & -5 & -3 \\ 0 & 3 & -1 & 5 & 15 & 1 \end{array}\right) R_2 \to -R_2 \quad \Rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 0 & 2 & 5 & -3 \\ 0 & 3 & -1 & 5 & 15 & 1 \end{array}\right) \begin{array}{l} R_1 \to R_2 + R_1 \\ \\ R_3 \to -3R_2 + R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 2 & 5 & 3 \\ 0 & 0 & -1 & -1 & 0 & -8 \end{array}\right) R_3 \to -R_3 \quad \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 0 & 8 \end{array}\right) R_3 \to -R_3$$

Therefore $B^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$.

(c)

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & -2 & -4 & 0 & 1 & 0 \\ -5 & 3 & 6 & 0 & 0 & 1 \end{array}\right) R_1 \Leftrightarrow R_2 \quad \Rightarrow \left(\begin{array}{ccc|ccc} 1 & -2 & -4 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \\ -5 & 3 & 6 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 \to -3R_1 + R_2 \\ R_3 \to 5R_1 + R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -4 & 0 & 1 & 0 \\ 0 & 7 & 14 & 1 & -3 & 0 \\ 0 & -7 & -14 & 0 & 5 & 1 \end{array}\right) R_3 \to R_2 + R_3 \quad \Rightarrow \left(\begin{array}{ccc|ccc} 1 & -2 & -4 & 0 & 1 & 0 \\ 0 & 7 & 14 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array}\right)$$

Since the left block has a row of zeros, it can not become identity matrix; which means the matrix C is not invertible.

2. Use properties of determinant to prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \end{vmatrix} = (b-a)(c-a)(c-b)\,.$$

Solution:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2+2a+1 & b^2+2b+1 & c^2+2c+1 \end{vmatrix} R_3 \to -R_1 + R_3$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2+2a & b^2+2b & c^2+2c \end{vmatrix} R_3 \to -2R_2 + R_3 \qquad = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \begin{array}{l} R_2 \to -aR_1 + R_3 \\ R_3 \to -a^2R_1 + R_2 \end{array}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} R_3 \to -(b+a)R_2 + R_3 \qquad = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2-(b^2-a^2) & c^2-a^2-(b+a)(c-a) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{vmatrix} = (b-a)(c-a)(c-b)\,.$$

3. Find the value of x such that

$$\begin{vmatrix} x & 2 & 1 \\ -1 & 0 & 1 \\ 0 & 3 & x \end{vmatrix} = \begin{vmatrix} 0 & x & -1 \\ 2 & 3 & 4 \\ 0 & 1 & -2 \end{vmatrix}.$$

Solution: Expansion of the determinants along the first column gives

$$x\begin{vmatrix} 0 & 1 \\ 3 & x \end{vmatrix} - (-1)\begin{vmatrix} 2 & 1 \\ 3 & x \end{vmatrix} = -2\begin{vmatrix} x & -1 \\ 1 & -2 \end{vmatrix}$$

$$x(0-3) + 1(2x-3) = -2(-2x+1) \quad \Rightarrow \quad -5x = 1 \quad \Rightarrow \quad x = -\frac{1}{5}\,.$$

4. Let A, B and C be 5×5 matrices such that $det(A) = 3$, $det(B) = -2$ and $det(C) = 10$. Evaluate each of the following:

(a) $det(AB^TC)$ **,**

Solution:

$$\begin{aligned} det(AB^TC) &= det(A)det(B^T)det(C) = det(A)det(B)det(C) \quad \textbf{(because } det(B^T) = det(B) \textbf{)} \\ &= (3)(-2)(10) \\ &= -60\,. \end{aligned}$$

(b) $det(A^2\,(B^T)^{-1})$ **,**

Solution:

$$\begin{aligned} det(A^2\,(B^T)^{-1}) &= det(A^2)det((B^T)^{-1}) \\ &= (det(A))^2\,det((B^T)^{-1}) \quad \big(\textbf{ because in general } det(A^n) = (det(A))^n\,\big) \\ &= (det(A))^2\,(\frac{1}{det(B^T)}) \quad \big(\textbf{ because } det((B^T)^{-1}) = \frac{1}{det(B^T)}\big) \\ &= (det(A))^2\,(\frac{1}{det(B)}) \quad \big(\textbf{ because } det(B^T) = det(B)\big) \\ &= (3)^2\,(\frac{1}{-2}) = -\frac{9}{2}\,. \end{aligned}$$

(c) $det(A^{-1}DB^{-3}D^{-1})$ **, (where** D **is another** 5×5 **matrix) .**

Solution:

$$\begin{aligned} det(A^{-1}DB^{-3}D^{-1}) &= det(A^{-1})det(D)det(B^{-3})det(D^{-1}) \\ &= \frac{1}{det(A)}det(D)\,(det(B^{-1}))^3\,\frac{1}{det(D)} \\ &= \frac{1}{det(A)}(\frac{1}{det(B)})^3\,det(D)\frac{1}{det(D)} \\ &= \frac{1}{det(A)}(\frac{1}{det(B)})^3 \\ &= \frac{1}{3}(\frac{1}{-2})^3 = -\frac{1}{24}\,. \end{aligned}$$

5. Find all values of x **for which the matrix** $A = \begin{pmatrix} x & 1-x & 3 \\ 1 & x & -1 \\ 2 & 1 & 1 \end{pmatrix}$ **is singular (that is not invertible).**

Solution: A **is singular if and only if** $det(A) = 0$**. But**

$$\begin{aligned} det(A) &= \begin{vmatrix} x & 1-x & 3 \\ 1 & x & -1 \\ 2 & 1 & 1 \end{vmatrix} = x\begin{vmatrix} x & -1 \\ 1 & 1 \end{vmatrix} - (1-x)\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & x \\ 2 & 1 \end{vmatrix} \\ &= x(x+1) - (1-x)(1+2) + 3(1-2x) \\ &= x^2 - 2x\,. \end{aligned}$$

So $det(A) = 0 \;\Rightarrow\; x^2 - 2x = 0 \;\Rightarrow\; x(x-2) = 0 \;\Rightarrow\; x = 0$ **or** $x = 2$**.**

6. Let $A = \begin{pmatrix} 2 & 4 & 1 & 1 \\ 2 & 4 & 1 & 2 \\ 2 & 4 & 2 & 3 \\ 2 & 5 & 0 & 1 \end{pmatrix}$**. Find $|A|$ by using properties of determinants to create an upper triangular matrix. For each elementary operation you use, explain what effect it has on determinant.**

Solution: First we notice that in a determinant
(i) if we switch two rows then the determinant will change sign;
(ii) if we replace one row by "a multiple of another row added to that row" then the determinant will not change.

$$|A| = \begin{vmatrix} 2 & 4 & 1 & 1 \\ 2 & 4 & 1 & 2 \\ 2 & 4 & 2 & 3 \\ 2 & 5 & 0 & 1 \end{vmatrix} \begin{matrix} \\ R_2 \to -R_1 + R_2 \\ R_3 \to -R_1 + R_3 \\ R_4 \to -R_1 + R_4 \end{matrix} = \begin{vmatrix} 2 & 4 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \end{vmatrix} R_2 \Leftrightarrow R_4$$

$$= - \begin{vmatrix} 2 & 4 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -\big((2)(1)(1)(1)\big) = -2$$

7. Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 3 & -1 \\ 2 & 1 & 4 \end{pmatrix}$**. First find A^{-1} then find all solutions of each of the following systems:**

(a) $A\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, **(b)** $(-3A)\mathbf{x} = \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix}$, **(c)** $A^{-1}\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, **(d)** $A^T\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

Solution:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right) R_3 \to -2R_1 + R_3 \quad \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right) R_2 \to 2R_2 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 2 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right) \begin{matrix} R_1 \to -R_2 + R_1 \\ \\ R_3 \to R_2 + R_3 \end{matrix} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 5 & -1 & -2 \\ 0 & 1 & -1 & -4 & 1 & 2 \\ 0 & 0 & -1 & -6 & 1 & 3 \end{array}\right) R_3 \to -R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 5 & -1 & -2 \\ 0 & 1 & -1 & -4 & 1 & 2 \\ 0 & 0 & 1 & 6 & -1 & -3 \end{array}\right) \begin{matrix} R_1 \to -3R_3 + R_1 \\ R_2 \to R_3 + R_2 \\ \\ \end{matrix} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -13 & 2 & 7 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 6 & -1 & -3 \end{array}\right)$$

Therefore $A^{-1} = \begin{pmatrix} -13 & 2 & 7 \\ 2 & 0 & -1 \\ 6 & -1 & -3 \end{pmatrix}$**.**

(a)

$$A\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \Rightarrow \mathbf{x} = A^{-1} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -13 & 2 & 7 \\ 2 & 0 & -1 \\ 6 & -1 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -7 \\ 1 \\ 4 \end{pmatrix}$$

Therefore $x = -7$**,** $y = 1$**, and** $z = 4$**.**

(b)

$$(-3A)\mathbf{x} = \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} \;\Rightarrow\; A\mathbf{x} = -\frac{1}{3}\begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} \;\Rightarrow\; A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \;\Rightarrow$$

$$\mathbf{x} = A^{-1}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \;\Rightarrow\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -13 & 2 & 7 \\ 2 & 0 & -1 \\ 6 & -1 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -27 \\ 4 \\ 12 \end{pmatrix}$$

Therefore $x = -27$**,** $y = 4$**, and** $z = 12$**.**

(c)

$$A^{-1}\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \;\Rightarrow\; \mathbf{x} = (A^{-1})^{-1}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \;\Rightarrow\; \mathbf{x} = A\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \;\Rightarrow$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 3 & -1 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 7 \end{pmatrix}$$

Therefore $x = 3$**,** $y = -4$**, and** $z = 7$**.**

(d)

$$A^T\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \;\Rightarrow\; \mathbf{x} = (A^T)^{-1}\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \;\Rightarrow\; \mathbf{x} = (A^{-1})^T\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \;\Rightarrow$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -13 & 2 & 7 \\ 2 & 0 & -1 \\ 6 & -1 & -3 \end{pmatrix}^T\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -13 & 2 & 6 \\ 2 & 0 & -1 \\ 7 & -1 & -3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -37 \\ 6 \\ 20 \end{pmatrix}$$

Therefore $x = -37$**,** $y = 6$**, and** $z = 20$**.**

8. Let $A = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 5 & -2 \\ 0 & 0 & -3 & 0 \end{pmatrix}$**.**

(a) **Find** $det(A)$**.**

Solution:

$$|A| = \begin{vmatrix} 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 5 & -2 \\ 0 & 0 & -3 & 0 \end{vmatrix} = -3\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -3 \end{vmatrix} = -3\big(2(2)(-3)\big) = 36 \neq 0$$

(b) **If** $adj(A) = \begin{pmatrix} 0 & 18 & 0 & 0 \\ 12 & 0 & b & 30 \\ 0 & 0 & 0 & -12 \\ a & 0 & 0 & 0 \end{pmatrix}$**, find the values of** a **and** b**.**

Solution:

$$a = c_{14} = (-1)^{1+4}\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -3 \end{vmatrix} = (-1)\big(2(2)(-3)\big) = 12\,,$$

$$b = c_{32} = (-1)^{3+2}\begin{vmatrix} 0 & 0 & 3 \\ 2 & 0 & 0 \\ 0 & -3 & 0 \end{vmatrix} = (-1)(3\begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix}) = (-1)(3)(-6) = 18\,.$$

(c) **Find A^{-1}.**

Solution:

$$A^{-1} = \frac{1}{|A|}\, adj(A) = \frac{1}{36}\begin{pmatrix} 0 & 18 & 0 & 0 \\ 12 & 0 & 18 & 30 \\ 0 & 0 & 0 & -12 \\ 12 & 0 & 0 & 0 \end{pmatrix}.$$

9. Let $\mathbf{u} = \langle 1, 3, -1\rangle$, $\mathbf{v} = \langle 0, 2, 2\rangle$ and $\mathbf{w} = \langle -1, 1, 4\rangle$. Is $\mathbf{r} = \langle 3, 4, 6\rangle$ a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$? If yes write $\mathbf{r}$ in terms of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$; if no explain why?

Solution: If there exist scalars c_1, c_2, and c_3 (at least one of them nonzero) such that $\mathbf{r} = c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}$, then $\mathbf{r}$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

$$c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{r} \;\Rightarrow\; c_1\langle 1, 3, -1\rangle + c_2\langle 0, 2, 2\rangle + c_3\langle -1, 1, 4\rangle = \langle 3, 4, 6\rangle \;\Rightarrow$$

$$\langle c_1 - c_3\,,\; 3c_1 + 2c_2 + c_3\,,\; -c_1 + 2c_2 + 4c_3\rangle = \langle 3, 4, 6\rangle \;\Rightarrow$$

$$\begin{aligned} c_1 - c_3 &= 3 \\ 3c_1 + 2c_2 + c_3 &= 4 \;\Rightarrow \\ -c_1 + 2c_2 + 4c_3 &= 6 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 3 & 2 & 1 & 4 \\ -1 & 2 & 4 & 6 \end{array}\right) \begin{array}{c} R_2 \to -3R_1 + R_2 \\ R_3 \to R_1 + R_3 \end{array} \;\Rightarrow\; \left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 2 & 4 & -5 \\ 0 & 2 & 3 & 9 \end{array}\right) R_3 \to -R_2 + R_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 2 & 4 & -5 \\ 0 & 0 & -1 & 14 \end{array}\right) \begin{array}{c} R_2 \to \frac{1}{2}R_2 \\ R_3 \to -R_3 \end{array} \;\Rightarrow\; \left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -\frac{5}{2} \\ 0 & 0 & 1 & -14 \end{array}\right) \begin{array}{c} R_1 \to R_3 + R_1 \\ R_2 \to -2R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & \frac{51}{2} \\ 0 & 0 & 1 & -14 \end{array}\right) RREF \;\Rightarrow\; c_1 = -11\,,\; c_2 = \frac{51}{2}\,,\; c_3 = -14\,.$$

Therefore yes $\mathbf{r} = -11\mathbf{u} + \frac{51}{2}\mathbf{v} - 14\mathbf{w}$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

10. For each of the following parts determine if the given vectors are linearly dependent or linearly independent. Show your work.

(a) $\mathbf{u} = \langle 3, 1, 3\rangle$, $\mathbf{v} = \langle 1, -4, 14\rangle$, $\mathbf{w} = \langle 4, 5, -7\rangle$,

Solution: Three vectors in $\mathbf{R}^3$ are linearly dependent if and only if the determinant of the matrix, whose columns are those three vectors, is zero.

$$\begin{aligned} \begin{vmatrix} 3 & 1 & 4 \\ 1 & -4 & 5 \\ 3 & 14 & -7 \end{vmatrix} &= 3\begin{vmatrix} -4 & 5 \\ 14 & -7 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 5 \\ 3 & -7 \end{vmatrix} + 4\begin{vmatrix} 1 & -4 \\ 3 & -4 \end{vmatrix} \\ &= 3(28 - 70) - 1(-7 - 15)) + 4(14 + 12) \\ &= -126 + 22 + 104 \\ &= 0\,. \end{aligned}$$

Therefore they are linearly dependent.

(b) $\mathbf{u} = \langle 4, 7\rangle$, $\mathbf{v} = \langle 8, 11\rangle$, $\mathbf{w} = \langle 12, 13\rangle$,

Solution: In general m vectors each with n component, when $m > n$, are linearly dependent. Here $m = 3 > n = 2$ so they are linearly dependent.

(c) $\mathbf{u} = \langle 1, 0, 2, 1\rangle$, $\mathbf{v} = \langle 3, -1, 1, 4\rangle$, $\mathbf{w} = \langle 4, -1, 3, 0\rangle$,

Solution: If $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = 0$, then,

$$c_1\langle 1, 0, 2, 1\rangle + c_2\langle 3, -1, 1, 4\rangle + c_3\langle 4, -1, 3, 0\rangle \;=\; \langle 0, 0, 0, 0\rangle \quad\Rightarrow$$

$$\langle c_1 + 3c_2 + 4c_3\,,\, -c_2 - c_3\,,\, 2c_1 + c_2 + 3c_3\,,\, c_1 + 4c_2\rangle \;=\; \langle 0, 0, 0, 0\rangle \quad\Rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0\\ 0 & -1 & -1 & 0\\ 2 & 1 & 3 & 0\\ 1 & 4 & 0 & 0 \end{array}\right) \begin{array}{c} \\ R_2 \to -R_2 \\ R_3 \to -2R_1 + R_3 \\ R_4 \to -R_1 + R_4 \end{array} \quad\Rightarrow\quad \left(\begin{array}{ccc|c} 1 & 3 & 4 & 0\\ 0 & 1 & 1 & 0\\ 0 & -5 & -5 & 0\\ 0 & 1 & -4 & 0 \end{array}\right) \begin{array}{c} R_1 \to -3R_2 + R_1 \\ \\ R_3 \to 5R_2 + R_3 \\ R_4 \to -R_2 + R_4 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -5 & 0 \end{array}\right) R_3 \Leftrightarrow R_4 \quad\Rightarrow\quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & -5 & 0\\ 0 & 0 & 0 & 0 \end{array}\right) R_3 \to -\tfrac{1}{5}R_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} R_1 \to -R_3 + R_1 \\ R_2 \to -R_3 + R_2 \\ \\ \\ \end{array} \quad\Rightarrow\quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{array}\right) RREF$$

Therefore $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, that is they are linearly independent.

(d) $\mathbf{u} = \langle -\frac{1}{2}, 2, \frac{3}{4}, 4\rangle$, $\mathbf{v} = \langle 1, -4, -\frac{3}{2}, -8\rangle$, $\mathbf{w} = \langle 5, -7, 1, 1\rangle$.

Solution: We solve it in two different ways. First solution is by inspection. Since $\mathbf{v} = -2\mathbf{u} + 0\mathbf{w}$ so they are linearly dependent.
For a second solution, if $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = 0$, then,

$$c_1\langle -\frac{1}{2}, 2, \frac{3}{4}, 4\rangle + c_2\langle 1, -4, -\frac{3}{2}, -8\rangle + c_3\langle 5, -7, 1, 1\rangle \;=\; \langle 0, 0, 0, 0\rangle \quad\Rightarrow$$

$$\langle -\frac{1}{2}c_1 + c_2 + 5c_3\,,\; 2c_1 - 4c_2 - 7c_3\,,\; \frac{3}{4}c_1 - \frac{3}{2}c_2 + c_3\,,\; 4c_1 - 8c_2 + c_3\rangle \;=\; \langle 0, 0, 0, 0\rangle \quad\Rightarrow$$

$$\begin{aligned} -\frac{1}{2}c_1 + c_2 + 5c_3 &= 0\\ 2c_1 - 4c_2 - 7c_3 &= 0\\ \frac{3}{4}c_1 - \frac{3}{2}c_2 + c_3 &= 0\\ 4c_1 - 8c_2 + c_3 &= 0 \end{aligned} \quad\Rightarrow$$

$$\left(\begin{array}{ccc|c} -\frac{1}{2} & 1 & 5 & 0\\ 2 & -4 & -7 & 0\\ \frac{3}{4} & -\frac{3}{1} & 1 & 0\\ 4 & -8 & 1 & 0 \end{array}\right) \begin{array}{c} R_1 \to -2R_1 \\ \\ \\ \\ \end{array} \quad\Rightarrow\quad \left(\begin{array}{ccc|c} 1 & -2 & -10 & 0\\ 2 & -4 & -7 & 0\\ \frac{3}{4} & -\frac{3}{2} & 1 & 0\\ 4 & -8 & 1 & 0 \end{array}\right) \begin{array}{c} \\ R_2 \to -2R_1 + R_2 \\ R_3 \to -\frac{3}{4}R_1 + R_3 \\ R_4 \to -4R_1 + R_4 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & -10 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 41 & 0 \end{array}\right) R_2 \to \tfrac{1}{13}R_2 \quad \Rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -10 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 41 & 0 \end{array}\right) \begin{array}{l} R_1 \to 10R_2 + R_1 \\ \\ R_3 \to -\frac{1}{2}R_2 + R_3 \\ R_4 \to -41R_2 + R_4 \end{array}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) RREF \quad \Rightarrow \begin{array}{c} c_1 - 2c_2 = 0 \\ c_3 = 0 \end{array} \quad \Rightarrow \begin{array}{c} c_1 = 2c_2 \\ c_3 = 0 \end{array}$$

Now let $c_2 = t$ then $c_1 = 2t$ and $c_3 = 0$. So the three vectors are linearly dependent. In particular if you choose $t = 1$ then $c_1 = 2$, $c_2 = 1$, $c_3 = 0$, that is $2\mathbf{u} + \mathbf{v} + 0\mathbf{w} = 0$ which means $\mathbf{v} = -2\mathbf{u}$ as wee saw in the first solution.

11. Given that $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are linearly independent, prove that $2\mathbf{u} + \mathbf{v}, 3\mathbf{v} - \mathbf{u}$ and $2\mathbf{v} + \mathbf{w}$ are also linearly independent.

Solution: Since $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are linearly independent, so any scalars k_1, k_2, and k_3 if $k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} = 0$, then $k_1 = 0, k_2 = 0, k_3 = 0$. Now if $c_1(2\mathbf{u}+\mathbf{v}) + c_2(3\mathbf{v}-\mathbf{u}) + c_3(2\mathbf{v}+\mathbf{w}) = 0$, then

$$2c_1\mathbf{u} + c_1\mathbf{v} + 3c_2\mathbf{v} - c_2\mathbf{u} + 2c_3\mathbf{v} + c_3\mathbf{w} = 0 \Rightarrow (2c_1 - c_2)\mathbf{u} + (c_1 + 3c_2 + 2c_3)\mathbf{v} + c_3\mathbf{w} = 0.$$

But since $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are linearly independent must

$$\begin{aligned} 2c_1 - c_2 &= 0 \\ c_1 + 3c_2 + 2c_3 &= 0 \\ c_3 &= 0 \end{aligned} \Rightarrow \begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since $det \begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix} = (1)det \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = 7 \neq 0$**, so the homogeneous system has a unique solution which is**

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

that is $c_1 = 0, c_2 = 0, c_3 = 0$. Therefore $2\mathbf{u}+\mathbf{v}$, $3\mathbf{v}-\mathbf{u}$ and $2\mathbf{v}+\mathbf{w}$ are linearly independent.