UNIVERSITY OF MANITOBA

DATE: February 27, 2013 MIDTERM
TITLE PAGE
EXAMINATION: Techniques of Classical and Linear Algebra TIME: 1 hour
COURSE: MATH 1210 EXAMINER: Borgersen, Harland, Moghaddam

NAME: **Solutions** ______________________

STUDENT NUMBER: ______________

SEAT NUMBER: ______________

SIGNATURE: (in ink) ______________________
(I understand that cheating is a serious offense)

- ❑ A01 9:30–10:20 AM MWF (100 St. Paul) N. Harland
- ❑ A02 1:30–2:20 PM MWF (221 Wallace) G. I. Moghaddam
- ❑ A03 9:30–10:20 AM MWF (136 Art Lab) R. Borgersen

INSTRUCTIONS TO STUDENTS:

This is a 1 hour exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 8 pages of questions and also 1 blank page for rough work. Please check that you have all the pages. You may remove the blank page if you want, but be careful not to loosen the staple.

The value of each question is indicated in the left hand margin beside the statement of the question. The total value of all questions is 70 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	8	
2	6	
3	7	
4	7	
5	15	
6	6	
7	9	
8	12	
Total:	70	

[8] 1. *Use mathematical induction on positive integer $n \geq 2$ to prove that*

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

Solution Let $P(n)$ stand for the formula

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

We start by showing that $P(n)$ is true for the first value $n = 2$. The left hand side is

$$\left(1-\frac{1}{2^2}\right)=1-\frac{1}{4}=\frac{3}{4}.$$

The right hand side is

$$\frac{2+1}{2(2)}=\frac{3}{4}.$$

Since the left hand side is equal to the right hand side, P(n) is true for $n = 2$.

Suppose that $P(n)$ is true for $n = k \geq 2$, that is

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}.$$

(Call the previous statement (*))We will show $P(n)$ is true for $n = k+1$, that is

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)=\frac{(k+1)+1}{2(k+1)}=\frac{k+2}{2k+2}.$$

Starting with the left hand side we get

$$\begin{aligned}
LHS &= \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)\\
&= \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)\\
&= \left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right) \qquad \text{by } (*)\\
&= \left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)\\
&= \frac{k^2+2k}{2k(k+1)}\\
&= \frac{k+2}{2(k+1)}\\
&= RHS
\end{aligned}$$

Therefore $P(n)$ is true for $n = k + 1$.

By the principle of mathematical induction $P(n)$ is true for all positive integers $n \geq 2$.

[6] 2. *Use the formula* $\sum_{k=1}^{m} k^2 = \frac{1}{6}\left[m(m+1)(2m+1)\right]$ *to evaluate* $\sum_{j=-5}^{4}\left[(j+7)^2-2(j+6)\right]$.

Solution Using a substitution $i = j + 6$ we can change the indices of summation from $j = -5$ to 4 to $i = 1$ to 10. This allows us to use the given formula. Rewriting the substitution yields $j = i - 6$ and therefore the summation becomes

$$\begin{aligned}
Sum &= \sum_{j=-5}^{4}\left[(j+7)^2 - 2(j+6)\right] \\
&= \sum_{i=1}^{10}\left[(i-6+7)^2 - 2(i-6+6)\right] \\
&= \sum_{i=1}^{10}\left[(i-6+7)^2 - 2(i-6+6)\right] \\
&= \sum_{i=1}^{10}\left[(i+1)^2 - 2i\right] \\
&= \sum_{i=1}^{10}\left[i^2 + 2i + 1 - 2i\right] \\
&= \sum_{i=1}^{10}(i^2+1) \\
&= \sum_{i=1}^{10} i^2 + \sum_{i=1}^{10} 1 \\
&= \frac{10 \cdot 11 \cdot 21}{6} + 10 \\
&= 385 + 10 \\
&= 395.
\end{aligned}$$

[7] 3. *Find, in simplified form, the Cartesian form of* $(\sqrt{6}+\sqrt{2}\,i)^{10}$.

Solution

Let $z=\sqrt{6}+\sqrt{2}\,i$. We start by changing z into exponential form. First we find the modulus of z to be

$$|z|=\sqrt{(\sqrt{6})^2+(\sqrt{2})^2}=\sqrt{6+2}=\sqrt{8}.$$

Then is $\theta = arg(z)$ we get

$$\tan\theta=\frac{\sqrt{2}}{\sqrt{6}}=\frac{1}{\sqrt{3}}\Rightarrow\theta=\frac{\pi}{6}$$

since $x, y > 0$ implies we get the angle in the first quadrant. Therefore in exponential form

$$z=\sqrt{8}e^{\pi i/6}.$$

Hence

$$\begin{aligned}
z^{10} &= \left(\sqrt{8}e^{\pi i/6}\right)^{10}\\
&= 8^5 e^{5\pi i/3}\\
&= 8^5\left(\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)\right)\\
&= 8^5\left(\frac{1}{2}+i\left(-\frac{\sqrt{3}}{2}\right)\right)\\
&= \frac{8^5}{2}-\frac{8^5\sqrt{3}}{2}\,i \qquad\qquad \text{or}\\
&= 2^{14}-2^{14}\sqrt{3}\,i.
\end{aligned}$$

[7] 4. *Find, in simplified form, the complex number z that satisfies the equation*

$$(1-i)^2(1+i)^2 \,+\, \overline{(4-3i)}\, z \,=\, 1+4i\,.$$

Solution

$(1-i)^2(1+i)^2$ can be evaluated in a couple ways. Directly we can get

$$(1-i)(1-i)(1+i)(1+i) = (1-2i+i^2)(1+2i+i^2) = (1-2i-1)(1+2i-1) = -4i^2 = 4$$

or we can be sneakier and get

$$\big((1-i)(1+i)\big)^2 = (1-i^2)^2 = (1+1)^2 = 4.$$

Either way the equation becomes

$$\begin{aligned} 4 + \overline{(4-3i)}\, z \,&=\, 1+4i \\ \Rightarrow (4+3i)\, z \,&=\, -3+4i \\ \Rightarrow z \,&=\, \frac{-3+4i}{4+3i}. \end{aligned}$$

To put this into Cartesian form, we multiply the numerator and denominator by the complex conjugate of the denominator to get

$$\begin{aligned} z &= \frac{(-3+4i)(4-3i)}{(4+3i)(4-3i)} \\ &= \frac{-12+9i+16i-12i^2}{16-12i+12i-9i^2} \\ &= \frac{-12+9i+16i+12}{16-12i+12i+9} \\ &= \frac{25i}{25} \\ &= i. \end{aligned}$$

5. *Consider the polynomial equation of* $P(x) = 0$ *where*

$$P(x) = 3x^4 - 18x^3 + 28x^2 + 12x - 20\,.$$

[3] (a) *Use Descartes' Rule of Signs to find the possible number of positive and negative roots of* $P(x) = 0$.

Solution

$P(x)$ has 3 sign changes, and therefore there are either 3 or 1 positive zeros. Since $P(-x) = 3(-x)^4 - 18(-x)^3 + 28(-x)^2 + 12(-x) - 20 = 3x^4 + 18x^3 + 28x^2 - 12x - 20$ has one sign change, there is exactly one negative zero.

[3] (b) *What are the possible rational zeros of* $P(x) = 0$ *?*

Solution

The rational root theorem says any rational root p/q of $P(x)$ must have p dividing $a_0 = -20$ and q dividing $a_n = 3$. Hence p is one of $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10$ or ± 20 and q is one of ± 1 or ± 3. Putting the possible numerators and denominators together leaves us with the list of possible rational zeros to be

$$\pm 1, \pm\frac{1}{3}, \pm 2, \pm\frac{2}{3}, \pm 4, \pm\frac{4}{3}, \pm 5, \pm\frac{5}{3}, \pm 10, \pm\frac{10}{3}, \pm 20, \pm\frac{20}{3}.$$

[3] (c) *Use Bounds Theorem to eliminate some of the possible zeros you found in part (b).*

Solution

Any zero x (not just real or rational) of $P(x)$ must satisfy

$$|x| < \frac{M}{|a_n|} + 1.$$

Now M is the maximum of the absolute value of the coefficients (or modulus if they are complex) with is 28. Hence the Bound's Theorem tells us that

$$|x| < \frac{28}{3} + 1 = \frac{31}{3}$$

eliminating only ± 20 from being a rational zero possibility.

[6] (d) *Given that* $3 + i$ *is a root of* $P(x)$, *find all other roots.*

Solution

Since the coefficients of $P(x)$ are all real, any complex zeros of $P(x)$ will also have its conjugate be a zero as well. Hence $3 - i$ is a zero. The factor theorem tells us that $(x - (3 + i))$ and $(x - (3 - i))$ are both factors and hence

$$(x - (3 + i))(x - (3 - i)) = x^2 - 6x + 10$$

must divide $P(x)$. Long divison can then tell us that

$$P(x) = (x^2 - 6x + 10)(3x^2 - 2).$$

Solving $3x^2 - 2 = 0$ leads to the other two solutions

$$3x^2 = 2 \Rightarrow x^2 = \frac{2}{3} \Rightarrow x = \pm\sqrt{\frac{2}{3}}.$$

Therefore the 4 solutions to $P(x) = 0$ are

$$3 \pm i,\ \pm\sqrt{\frac{2}{3}}.$$

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[6] 6. *Let* $A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 0 & 1 & 4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & -1 \end{pmatrix}$. *Find a matrix* X *such that* $AB^T - X = I$.

Solution

Some simple manipulations leads to

$$X = AB^T - I$$

where B^T is the transpose which is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 \\ 1 & 4 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$$

Then we can find

$$AB^T = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 4 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 3 & 0 \end{pmatrix}.$$

Hence

$$X = AB^T - I = \begin{pmatrix} 3 & 8 \\ 3 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 3 & -1 \end{pmatrix}$$

7. *For $a > 0$ let $\mathbf{u} = \langle 2a, -a, a\rangle$, $\mathbf{v} = \langle 1, 1, 2\rangle$ and $\mathbf{w} = \langle 3, 4, 1\rangle$*

[5] (a) *Find the angle between $\mathbf{u}$ and $\mathbf{v}$.*

Solution

From $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ we can get

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} \\
&= \frac{\langle 2a, -a, a\rangle \cdot \langle 1, 1, 2\rangle}{|\langle 2a, -a, a\rangle|\,|\langle 1, 1, 2\rangle|} \\
&= \frac{2a + (-a) + 2a}{\sqrt{(2a)^2 + (-a)^2 + a^2}\sqrt{1^2 + 1^2 + 2^2}} \\
&= \frac{3a}{\sqrt{36a^2}} \\
&= \frac{3a}{6a} \qquad\qquad \text{since } a > 0. \\
&= \frac{1}{2}.
\end{aligned}$$

Therefore $\theta = \dfrac{\pi}{3}$.

[4] (b) *Find the value of a for which $\mathbf{u} + \mathbf{v}$ and $\mathbf{w}$ are perpendicular.*

Solution

$\mathbf{u} + \mathbf{v}$ and $\mathbf{w}$ are perpendicular if $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = 0$. Hence we are solving for a where

$$\begin{aligned}
& (\langle 2a, -a, a\rangle + \langle 1, 1, 2\rangle) \cdot \langle 3, 4, 1\rangle = 0 \\
\Rightarrow\quad & \langle 2a + 1, -a + 1, a + 2\rangle \cdot \langle 3, 4, 1\rangle = 0 \\
\Rightarrow\quad & (2a + 1)(3) + (-a + 1)(4) + (a + 2)(1) = 0 \\
\Rightarrow\quad & 6a + 3 - 4a + 4 + a + 2 = 0 \\
\Rightarrow\quad & 3a + 9 = 0 \\
\Rightarrow\quad & a = -3.
\end{aligned}$$

8. *Consider the point* $P(2,3,5)$ *and the two planes* $\Pi_1 : \; 2x+3y-z=0$ *and* $\Pi_2 : \; -x+y+z=0$.

[2] (a) *Is the point* P *on the plane* Π_1 *?*

Solution

The point is on the plane if $x=2, y=3, z=5$ satifies the equation. However since

$$2(2)+3(3)-5=8\neq 0$$

the point P is not on the plane Π_1.

[4] (b) *Show that the two planes* Π_1 *and* Π_2 *are perpendicular.*

Solution

Two planes are perpendicular if and only if their normal vectors are perpendicular which we can show by showing their scalar (dot) product is 0. A normal vector to Π_1 is $\mathbf{n_1}=\langle 2,3,-1\rangle$ and a normal vector to Π_2 is $\mathbf{n_2}=\langle -1,1,1\rangle$. Since

$$\mathbf{n_1}\cdot\mathbf{n_2}=\langle 2,3,-1\rangle\cdot\langle -1,1,1\rangle=2(-1)+3(1)-1(1)=0$$

we know the planes are perpendicular.

[6] (c) *Find parametric equations of the line through the point* P *and parallel to the line* $2x+3y-z=0\,,\;\; -x+y+z=0\,.$

Solution

To find a line, we need a point on the line which we have as $P(2,3,5)$ and a vector $\mathbf{v}$ parallel to the line. Since the line is parallel to the line of intersection of Π_1 and Π_2 it must be perpendicular to both $\mathbf{n_1}$ and $\mathbf{n_2}$. Hence we can take $\mathbf{v}$ to be

$$\begin{aligned}
\mathbf{v} &= \langle 2,3,-1\rangle\times\langle -1,1,1\rangle\\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & 3 & -1\\ -1 & 1 & 1\end{vmatrix}\\
&= \begin{vmatrix} 3 & -1\\ 1 & 1\end{vmatrix}\hat{\mathbf{i}} - \begin{vmatrix} 2 & -1\\ -1 & 1\end{vmatrix}\hat{\mathbf{j}} + \begin{vmatrix} 2 & 3\\ -1 & 1\end{vmatrix}\hat{\mathbf{k}}\\
&= (3-(-1))\hat{\mathbf{i}} - (2-1)\hat{\mathbf{j}} + (2-(-3))\hat{\mathbf{k}}\\
&= 4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 5\hat{\mathbf{k}}.
\end{aligned}$$

Hence the parametric equations are

$$\begin{aligned}
x &= x_0 + v_x t = 2+4t\\
y &= y_0 + v_y t = 3-t\\
z &= z_0 + v_z t = 5+5t
\end{aligned}$$