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[8] 1. Use mathematical induction to prove that

$$\sum_{j=1}^{n} 3(j+1)(j-2) = n(n^2-7)\,,$$

for all integers $n \geq 1$.

Solution: Let $P(n)$ be the statement "$\sum_{j=1}^{n} 3(j+1)(j-2) = n(n^2-7)$".

We prove by induction on n that $P(n)$ holds for all $n \geq 1$.

$P(1)$ is the assertion "$3(1+1)(1-2) = 1(1^2-7)$", which is certainly true since both sides evaluate to -6.

Assume that $P(n)$ holds for some fixed number k: that is, that

$$\sum_{j=1}^{k} 3(j+1)(j-2) = k(k^2-7)\,.$$

We want to prove that $P(k+1)$ holds, that is, that

$$\sum_{j=1}^{k+1} 3(j+1)(j-2) = (k+1)((k+1)^2-7)\,.$$

The right hand side of this equation simplifies:

$$(k+1)((k+1)^2-7) = (k+1)(k^2+2k+6) = k^3+3k^2-4k-6\,.$$

For the left hand side,

$$\begin{aligned}
\sum_{j=1}^{k+1} 3(j+1)(j-2) &= \left(\sum_{j=1}^{k} 3(j+1)(j-2)\right) + 3((k+1)+1)((k+1)-2) \\
&= k(k^2-7) + 3(k+2)(k-1) \\
&\qquad\qquad \text{(by the induction hypothesis)} \\
&= k^3 - 7k + 3(k^2+k-2) \\
&= k^3 + 3k^2 - 4k - 6
\end{aligned}$$

Since the two sides of the equation are equal, $P(k+1)$ holds, and hence by the principal of mathematical induction, $P(n)$ holds for all $n \geq 1$.

2. Find, in simplified form, the Cartesian form of each of the following expressions:

[5] (a) $\dfrac{1-i}{i-\frac{1}{1-i}}$

Solution:

$$\begin{aligned}\frac{1-i}{i-\dfrac{1}{1-i}} &= \frac{1-i}{\dfrac{i(1-i)-1}{1-i}} = \frac{(1-i)^2}{i-i^2-1} \\ &= \frac{1-2i-1}{i} = \frac{-2i}{i} \\ &= -2\end{aligned}$$

Remark: if we want to make the Cartesian form explicit, we can write "$-2+0i$".

Remark: There are many different equally valid pathways through this bit of arithmetic!

[5] (b) $(\sqrt{2}-i\sqrt{6})^6$

Solution: Let $z=\sqrt{2}-i\sqrt{6}$. Then $|z|=\sqrt{2+6}=\sqrt{8}$ and $\arg(z)=\operatorname{Tan}^{-1}\left(\frac{-\sqrt{6}}{\sqrt{2}}\right)=\operatorname{Tan}^{-1}(-\sqrt{3})=-\frac{\pi}{3}$.
So $z^6=\sqrt{8}^6\mathbf{e}^{(-\frac{\pi}{3}i)6}=512\mathbf{e}^{-2\pi i}=512$.

[$\mathbf{e}^{2n\pi i}=1$ for any integer n]

Remark: if we want to make the Cartesian form explicit, we can write "$512+0i$".

3. Consider the polynomial equation of $P(x) = 0$ where

$$P(x) = 3\,x^4 + x^3 - 6\,x^2 - 5\,x - 1$$

[3] (a) Use the Rational Root Theorem to list all possible rational roots of $P(x)$.

Solution: Let $\dfrac{p}{q}$ be a rational root (in lowest form). Then p divides the constant term and q divides the coefficient of the highest degree term.
So the possible values of p are 1, -1 and the possible values of q are 1, -1, 3, -3
Therefore, the possible rational roots are $\pm 1\,,\ \pm\frac{1}{3}$.

[4] (b) Use direct substitution to check whether any of them are in fact roots.

Solution: We check all four possibilities given by part (a)!
$P(1) = 3 + 1 - 6 - 5 - 1 \neq 0$.
$P(-1) = 3 - 1 - 6 + 5 - 1 = 0$.
$P(\frac{1}{3}) = 3\left(\frac{1}{81}\right) + \frac{1}{27} - 6\left(\frac{1}{9}\right) - 5\left(\frac{1}{3}\right) - 1 \neq 0$.
$P(-\frac{1}{3}) = 3\left(\frac{1}{81}\right) - \frac{1}{27} - 6\left(\frac{1}{9}\right) + 5\left(\frac{1}{3}\right) - 1 = 0$.
Therefore -1 and $-\frac{1}{3}$ are roots of $P(x)$, and 1 and $\frac{1}{3}$ are not.

[7] (c) Use your results from (b) to find all the roots of $P(x)$.

Solution: Thus $x+1$ and $3x+1$ are factors of $P(x)$. $(x+1)(3x+1) = 3x^2+4x+1$, and by long division we find $P(x) = (3x^2+4x+1)(x^2-x-1)$.
By the quadratic formula, the roots of $x^2 - x - 1$ are $\dfrac{1 \pm \sqrt{5}}{2}$.
So all the roots of $P(x)$ are -1, $-\frac{1}{3}$, and $\dfrac{1 \pm \sqrt{5}}{2}$.

Solution: COMMENT: There was a lot of confusion between "roots" of a polynomial and "factors" of a polynomial.
A *root* of a polynomial $P(x)$ is a *number* c such that $P(c) = 0$. Roots of polynomials are sometimes also called *zeroes* of the polynomial.
A *factor* of a polynomial $P(x)$ is another polynomial $Q(x)$ which divides $P(x)$: $P(x) = Q(x)R(x)$ for some other polynomial $R(x)$.
In this problem, -1, $-\frac{1}{3}$, and $\dfrac{1 \pm \sqrt{5}}{2}$ are the roots of $P(x)$. Along the way we found two linear factors of $P(x)$, namely $x+1$ and $3x+1$, and an irreducible quadratic factor of $P(x)$, namely $x^2 - x - 1$. $P(x) = (x+1)(3x+1)(x^2 - x - 1)$.

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[4] 4. Let z denote any complex number. Prove that if $|z| = 1$, then $\overline{z} = \dfrac{1}{z}$.

Solution: [Version 1. This is the better of the two versions.]

$|z|^2 = z\overline{z}$, so if $|z| = 1$ then $z\overline{z} = 1$. Hence $\overline{z} = \dfrac{1}{z}$.

Solution: [Version 2: This is also acceptable.]

If $z = a + bi$, (a, b, real numbers) and $|z| = 1$, then $a^2 + b^2 = 1$.

$$\frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2} = a - bi = \overline{z}.$$

5. Consider the polynomial equation of $P(x) = 0$ where

$$P(x) = 3x^7 - 5x^6 - 11x^5 + 8x^3 + 7x^2 - 13x + 9$$

[4] (a) Use the Bounds Theorem to deduce a bound on $|z|$, where z denotes any root of $P(x)$.

Solution: The leading coefficient is 3. The maximum absolute value of any other coefficient is 13. Therefore, by the Bounds Theorem, if z is any root of $P(x)$,

$$|z| < \frac{13}{3} + 1 = \frac{16}{3}.$$

[4] (b) Use Descartes' Rule of Signs to find the possible numbers of positive and negative real roots.
(Your are **not** asked to find the roots of $P(x)$.)

Solution: $P(x)$ has 4 changes in sign of the coefficients, so $P(x)$ has 4, 2, or 0 positive real roots.

$P(-x) = -3x^7 - 5x^6 + 11x^5 - 8x^3 + 7x^2 + 13x + 9$, which has 3 changes in sign of the coefficients, so $P(x)$ has 3 or 1 negative real roots.

6. Let $A = \begin{pmatrix} 5 & -4 & 3 \\ -1 & 2 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$. Evaluate each of the following expressions or explain why it is not defined.

[6] (a) $AA^T - B^2$.

Solution:

$$AA^T = \begin{pmatrix} 5 & -4 & 3 \\ -1 & 2 & 7 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ -4 & 2 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 50 & 8 \\ 8 & 54 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 3 \\ 3 & 1 \end{pmatrix}$$

So

$$AA^T - B^2 = \begin{pmatrix} 40 & 5 \\ 5 & 53 \end{pmatrix}$$

[2] (b) $(2B - B^T)A^T$.

Solution: $(2B - B^T)$ is defined, with 2 columns, but A^T has 3 rows, so these two matrices are not compatible for multiplication.

7. Let $\mathbf{u} = \langle 2\,, -3\,, 5 \rangle$, $\mathbf{v} = \langle 1\,, 2\,, -3 \rangle$ and $\mathbf{w} = \langle k+4\,, k\,, 2k \rangle$.

[4] (a) Find the value of k for which $\mathbf{u}+\mathbf{v}$ and $\mathbf{w}$ are perpendicular.

Solution: $\mathbf{u}+\mathbf{v} = \langle 3, -1, 2 \rangle$, and so $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = 3(k+4) - k + 2(2k) = 6k+12$.
For $\mathbf{u}+\mathbf{v}$ and $\mathbf{w}$ to be perpendicular, their dot product must be 0.
Thus $k = -2$.

[4] (b) Find $(\mathbf{u}+\mathbf{v}) \times (\mathbf{u}-\mathbf{v})$.

Solution: From (a), $\mathbf{u}+\mathbf{v} = \langle 3, -1, 2 \rangle$, and we find $\mathbf{u}-\mathbf{v} = \langle 1, -5, 8 \rangle$. So

$$\begin{aligned}
(\mathbf{u}+\mathbf{v}) \times (\mathbf{u}-\mathbf{v}) &= \langle 3, -1, 2 \rangle \times \langle 1, -5, 8 \rangle \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & -1 & 2 \\ 1 & -5 & 8 \end{vmatrix} \\
&= \begin{vmatrix} -1 & 2 \\ -5 & 8 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 3 & 2 \\ 1 & 8 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 3 & -1 \\ 1 & -5 \end{vmatrix} \hat{\mathbf{k}} \\
&= 2\,\hat{\mathbf{i}} - 22\,\hat{\mathbf{j}} + (-14)\,\hat{\mathbf{k}} \\
&= \langle 2, -22, -14 \rangle
\end{aligned}$$

8. Consider the point $P(4,1,1)$, the plane $\Pi : 3x - y + 2z = -1$, and the line $\ell :\ x = 1 + t\,,\ y = -2 + 2t\,,\ z = t\,.$

[4] (a) Find the point of intersection of the line ℓ and the plane Π.

Solution: If a typical point $\langle 1+t,\ -2+2t,\ t\rangle$ on ℓ lies on Π then necessarily $3(1+t) - (-2+2t) + 2t = -1$, that is, $3t + 5 = -1$, so $t = -2$. Thus the point of intersection is $\langle -1,\ -6,\ -2\rangle$.

[4] (b) Find an equation of the plane which passes through the point P and is perpendicular to the line ℓ.

Solution: If a plane is perpendicular to ℓ, then a direction vector of ℓ is a normal vector to the plane. A direction vector of ℓ is given by the coefficients of the parameter in the parametric form of the equation of ℓ, so we can take $\mathbf{n} = \langle 1,\ 2,\ 1\rangle$ as a normal vector to the plane.
Thus an equation of the plane (point-normal form) is

$$\mathbf{n} \cdot (\mathbf{x} - \langle 4,\ 1,\ 1\rangle) = 0\,.$$

Equivalent forms are:

$$\langle 1,\ 2,\ 1\rangle \cdot (\langle x,\ y,\ z\rangle - \langle 4,\ 1,\ 1\rangle) = 0$$

$$1 \cdot (x-4) + 2 \cdot (y-1) + 1 \cdot (z-1) = 0$$

$$x + 2y + z - 7 = 0$$

$$x + 2y + z = 7$$