UNIVERSITY OF MANITOBA

FINAL EXAMINATION

DATE: April 15, 2016 TITLE PAGE

COURSE: MATH 1210 TIME: 3 hours

EXAMINATION: Classical and Linear Algebra EXAMINER: various

FAMILY NAME: (Print in ink) ______________________

GIVEN NAME(S): (Print in ink) ______________________

STUDENT NUMBER: ____________

SEAT NUMBER: ____________

SIGNATURE: (in ink) ______________________

(I understand that cheating is a serious offense)

Please indicate your instructor by placing a check mark in the appropriate box below.

- ❑ A01 9:30–10:20 207 Buller S. Portet
- ❑ A02 1:30–2:20 221 Wallace S. Tsaturian
- ❑ A03 9:30–10:20 205 Armes M. Davidson

INSTRUCTIONS TO STUDENTS:

This is a 3 hour exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page and 14 pages of questions. Please check that you have all the pages.

The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 130 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	53	
2	8	
3	8	
4	10	
5	8	
6	10	
7	6	
8	12	
9	6	
10	9	
Total:	130	

1. The following are short answer questions. Parts are not necessarily related.

[5] (a) Use the direct method to find the inverse of the matrix $A = \begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$.

[3] (b) Is the following set of vectors linearly dependent or linearly independent? Justify your answer.
$\{\langle 1, 1\rangle, \langle -3, 12\rangle, \langle 6, -7\rangle\}$

[4] (c) For what values of k is the polynomial $P(x) = x^4 + kx^3 - x^2 + x + 12$ divisible by $x + k$?

[5] (d) Is the following set of vectors linearly dependent or linearly independent? Justify your answer.

$\{\langle 1,1,2\rangle, \langle -3,1,2\rangle, \langle 6,-7,3\rangle\}$

[5] (e) Let T be the transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$ defined by $T(\mathbf{x}) = \mathbf{Ax}$ where

$$A = \begin{pmatrix} 1 & -1 & 2 & 5 \\ -1 & 3 & 4 & 2 \\ 2 & 4 & 5 & -7 \\ 5 & 2 & -7 & 2 \end{pmatrix}$$

i. Note that $A^T = A$. What is the term for matrices having this property?

ii. How many eigenvalues, including multiplicity, does T have? (Why?)

iii. Without computing the eigenvalues, state how many of the eigenvalues are real. (Why?)

[3] (f) Give in symmetric form an equation of the line that is perpendicular to the plane $2x - 3y + z = 15$ and passes through the point $P(1, 2, 3)$.

[4] (g) Suppose $a, b, c, d \in \mathbb{R}$ and that $3 - i$ and $2 + i$ are two of the zeros (over $\mathbb{C}$) of the polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$. Find all the other zeros of $p(x)$. What is the value of d?

[4] (h) For the following, let $\left[A|\mathbf{b}\right]$ be the augmented matrix of a system of 6 linear equations and 6 unknowns. For each blank entry in the following table, there is only one correct value given the other information in that row. Fill in the blanks, using 0 for 'no solutions', 1 for 'unique solution' and ∞ for 'infinitly many solutions'.

Rank of A	Rank of $\left[A\|\mathbf{b}\right]$	Number of solutions of $AX = \mathbf{b}$
	3	0
3	3	
	5	∞
3		0
4		∞
6		
5	6	

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[6] (i) For $z_1, z_2 \in \mathbb{C}$, show that $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$.

[4] (j) Fill in the blanks as appropriate:

For an $n \times n$ matrix A, the following are equivalent:

- A is invertible (A^{-1} exists).
- The reduced row echelon form of A is __________
- $\det(A)$ __________
- The rank of A is __________.
- $AX = \mathbf{0}$ has ________________ solution(s).

[6] (k) Use Cramer's rule to solve the following system of equations:

$$\begin{aligned} 4x_1 \;&+\; 3x_2 = -2 \\ 11x_1 \;&+\; 7x_2 = 3 \end{aligned}$$

[4] (l) Let T be the transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{pmatrix} 4 & 1 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}.$$

Which of the following vectors are eigenvectors of T? If the vector is an eigenvector, what is the associated eigenvalue?

$$v_1 = \begin{pmatrix} 7 \\ -7 \\ -7 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

[8] 2. Find all the roots of the polynomial

$$p(x) = x^4 + 3x^3 - 2x^2 - 10x - 12\,.$$

[8] 3. Find all solutions to $z^3 = -27$.
Give your answers in **both** Cartesian and exponential form.
Solutions in exponential form should have arguments in principal value.

[7] 4. (a) Write the vector $\langle 4, 7, -6 \rangle$ as a linear combination of $\langle 1, 2, -1 \rangle$, $\langle 1, 3, 2 \rangle$ and $\langle 2, 4, -1 \rangle$.

[3] (b) Suppose T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ where $T(\langle 1, 2, -1 \rangle) = \langle 0, 1, 1 \rangle$, $T(\langle 1, 3, 2 \rangle) = \langle 1, 0, 1 \rangle$, and $T(\langle 2, 4, -1 \rangle) = \langle 0, 1, 1 \rangle$. Find $T(\langle 4, 7, -6 \rangle)$.

5. We define two lines as follows:

$$\ell_1 : x = 3 + t, y = 4 - t, z = -1 + t\,;\, t \in \mathbb{R}$$

$$\ell_2 : x = 4 + s, y = -3 + 3s, z = -1 + 2s\,;\, s \in \mathbb{R}$$

[3] (a) Show that these lines do not have a point of intersection.

[5] (b) Find an equation of a plane that contains all of the points of ℓ_1 and none of the points of ℓ_2.

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[10] 6. Suppose $A = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$.

Use mathematical induction to show that $A^{2n} = 6^n I_2$ for all $n \geq 1$.

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7. Suppose $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ has determinant 2.

   What is the determinant of the following matrices?

[2] (a) $B_1 = \begin{pmatrix} d & e & f \\ 3a & 3b & 3c \\ g & h & i \end{pmatrix}$

[2] (b) $B_2 = \begin{pmatrix} a & b & c \\ a+2d & b+2e & c+2f \\ g-4a & h-4b & i-4c \end{pmatrix}$

[2] (c) $B_3 = \begin{pmatrix} a+d-g & b+e-h & c+f-i \\ d-g & e-h & f-i \\ g-d & h-e & i-f \end{pmatrix}$

[9] 8. (a) Use the adjoint method to find the inverse of the matrix

$$A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & -4 & -1 \\ -1 & 5 & -2 \end{pmatrix}.$$

[3] (b) Use the information from part (a) to find the solution to :

$$\begin{array}{rcrcrl} x_1 & - & 3x_2 & + & x_3 = & -2 \\ 2x_1 & - & 4x_2 & - & x_3 = & 3 \\ -x_1 & + & 5x_2 & - & 2x_3 = & 5 \end{array}$$

[6] 9. Let T be the transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by $T(\mathbf{x}) = A\mathbf{x}$ where

$A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 4 \\ 1 & -1 & -2 \end{pmatrix}$. Find all eigenvalues of T. Show your work.

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[9] 10. Let T be the transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{pmatrix} -2 & 1 & 2 \\ 1 & -2 & 2 \\ 1 & 1 & -1 \end{pmatrix}.$$

The eigenvalues of T are -3 and 1 (one of these values is a multiple root of the characteristic polynomial). Find all the eigenvectors of T.