UNIVERSITY OF MANITOBA
DEPARTMENT OF MATHEMATICS
MATH 1210 Techniques of Classical and Linear Algebra
FINAL EXAM
2017-04-29 8:00 - 11:00am

FAMILY / LAST NAME: (Print in ink) ______________________________

FIRST NAME: (Print in ink) ______________________________

STUDENT NUMBER: ______________________________

SIGNATURE: (in ink) ______________________________
(I understand that cheating is a serious offense)

INSTRUCTIONS TO STUDENTS:

This is a 3 hour exam.

Fill in all the information above.

***No** calculators, texts, notes, cell phones, pagers, translators or other electronics are permitted. No outside paper is permitted.*

This exam has a title page and 14 other pages, 2 of which can be used for rough work. You may remove the scrap pages if you like, but be careful not to loosen the staple. Please check that you have all pages.

Important: Any work you want marked must be written on the front of a page. If you need more room, write your answers on the scrap paper. Anything written on the back of pages will not be marked (so feel free to use these for rough work).

There are 13 questions for a total of 120 marks.
Show all your work clearly and justify your answers using complete sentences (unless it is explicitly stated that you do not have to do that). **Unjustified answers may receive LITTLE or NO CREDIT.**

If a question calls for a specific method, no credit will be given for other methods.

Short Answer Questions

1. The following are short answer questions. Parts are not necessarily related.

[5] (a) Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}.$$

Find the matrix C for which $A + C^T = 2B$.

[2] (b) Express the following complex number in exponential form. **Underline your final answer.** $z_1 = 1$

[2] (c) Is the set of vectors $\{\langle 0,1,0\rangle, \langle 1,1,-2\rangle, \langle 0,0,0\rangle\}$ linearly dependent? Justify your answer.

[3] (d) Consider T a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such as $T(\langle 1,0,0\rangle) = \langle 1,2,1\rangle$, $T(\langle 0,1,0\rangle) = \langle 2,2,1\rangle$ and $T(\langle 0,0,1\rangle) = \langle 1,0,1\rangle$. Find the image of $\langle 1,2,3\rangle$ by T.

[3] (e) Consider the linear transformation T associated with the matrix

$$A = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}.$$

What is the image of a vector $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$?

[2] (f) Find the eigenvalues of the matrix $A = \begin{bmatrix} -3 & 3 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

[5] (g) Consider the following matrix

$$A = \begin{bmatrix} -3 & 0 & 1 \\ 0 & -5 & -1 \\ 1 & -1 & -2 \end{bmatrix}.$$

The characteristic polynomial of the matrix A is $P(\lambda) = -\lambda^3 - 10\lambda^2 - 29\lambda - 22$. Without finding the eigenvalues, show that all the eigenvalues of A are real negative.

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[3] (h) Write the following sum using sigma notation with index starting at 1 (do not evaluate):
$\frac{1}{6}+\frac{1}{3}+\frac{2}{3}+\frac{4}{3}+\frac{8}{3}+\cdots+\frac{64}{3}$

[3] (i) Write the following sum using sigma notation with index starting at 1 (do not evaluate):
$-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}$

Let

$$A=\begin{bmatrix} -1 & -1 \\ \frac{2}{3} & 3 \end{bmatrix} \quad B=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix} \quad C=\begin{bmatrix} 4 & 3 \\ 3 & 3 \\ -1 & 3 \end{bmatrix}.$$

Perform the required operation, or explain why it is not possible.

[3] (j) B^2

[2] (k) $(AC^T)^T-(A+3I_2)$

Long Answer Questions

[12] **2.** Prove using mathematical induction that for every $n \geq 2$, $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}^n = 5^{n-1} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$.

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[4] **3.** Express the following complex number in exponential form: $z_2 = \dfrac{-\sqrt{15}}{\sqrt{2}} - i\dfrac{\sqrt{10}}{2}$

[4] **4.** Find a unit vector $\mathbf{v}$ that is orthogonal to $\langle -2, -1, 2\rangle$ and $\langle 1, -2, 0\rangle$.

[4] **5.** For what values of k is the polynomial $P(x) = x^4 + kx^3 + x^2 - x - 1$ divisible by $x + k$?

[5] **6.** Give bounds of solutions of

$$(2 - i)x^{210} - \sqrt{3}x^4 + (1 + i\sqrt{2})x^2 + i = 0$$

[5] **7.** Use Cramer's rule to solve the linear system

$$\begin{aligned} 2x - y &= 1 \\ 3x + y &= 4 \end{aligned}$$

[5] **8.** Consider the system and fill out the table (no justification necessary):

$$\begin{aligned} x + 2y - 3z &= 4 \\ -7y + 14z &= -10 \\ (a^2 - 16)z &= a - 4 \end{aligned}$$

The system will have	when a is equal to
No solution	
Exactly one solution	
Infinitely many solutions	

[10] **9.** (a) Consider the invertible matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix}$. Find A^{-1} using the adjoint method.

[3] (b) Consider the linear transformation T associated with the matrix A. Using part (a), find the vector $\mathbf{v}$ whose the image by the transformation T is the vector $\langle 1, 1, 1 \rangle$.

[2] (c) Using (a) and (b) above, write $\langle 1, 1, 1 \rangle$ as a linear combination of $\langle 1, 2, 1 \rangle$, $\langle 2, 1, 1 \rangle$ and $\langle 0, -1, -2 \rangle$.

[7] **10.** Find all complex numbers z such that $z^3 = e^{i\pi}$. Write all your final answers in **Cartesian form**. Show all your work.

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[7] **11.** Let $p(x) = x^4 - x^3 + 6x^2 + 14x - 20$. Given that $p(1 - 3i) = 0$, write $p(x)$ as a product of linear and irreducible real factors.

12. Consider the two planes

$$\Pi_1 : x + 2y + 3z = 6$$

$$\Pi_2 : -2x + y + z = 0.$$

[7] (a) Use Gauss-Jordan elimination to find the line of intersection of these two planes. Write your answer in parametric form.

[1] (b) Find a vector along the line of intersection.

[3] (c) Show that the vector found in part (b) is perpendicular to the normal vectors of both planes.

[8] **13.** Find the eigenpairs of the matrix A

$$A = \begin{bmatrix} 2 & 7 \\ -1 & -6 \end{bmatrix}.$$

END OF EXAM

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SCRAP PAPER