This assignment is **optional** and does not need to be handed in. Attempt all questions, write out nicely written solutions (showing all your work), and the solutions will be posted on Fri, Feb 3, 2017, at which point you can mark your own work. If you have any questions regarding differences between what you wrote and what the solution key says, please contact your professor. **At least one question from this assignment will be found on Quiz 1.**

1. Show that for all $n \geq 1$, $4^3 + 8^3 + \cdots + (4n)^3 = 16n^2(n+1)^2$.

Solution: For all $n \geq 1$, let P_n denote the statement that

$$4^3 + 8^3 + \cdots + (4n)^3 = 16n^2(n+1)^2.$$

Base Case. The statement P_1 says that $4^3 = 16(1)^2(1+1)^2 = 16(4) = 4^3$, which is true. Therefore P_1 holds.

Inductive Step. Fix $k \geq 1$ and assume that P_k holds, that is,

$$4^3 + 8^3 + \cdots + (4k)^3 = 16k^2(k+1)^2.$$

It remains to show that P_{k+1} holds, that is,

$$4^3 + 8^3 + \cdots + (4(k+1))^3 = 16(k+1)^2(k+2)^2.$$

$$\begin{aligned}
4^3 + 8^3 + \cdots + (4(k+1))^3 &= 4^3 + 8^3 + \cdots + (4k+4)^3 \\
&= 4^3 + 8^3 + \cdots + (4k)^3 + (4k+4)^3 \\
&= 16k^2(k+1)^2 + (4k+4)^3 \\
&= 16k^2(k+1)^2 + 4^3(k+1)^3 \\
&= \big(k^2 + 4(k+1)\big)\, 4^2(k+1)^2 \\
&= (k^2 + 4k + 4)4^2(k+1)^2 \\
&= 16(k+1)^2(k+2)^2.
\end{aligned}$$

Thus P_{k+1} holds.

Therefore, by PMI, for all $n \geq 1$, P_n holds.

2. (a) Write the sum $1 + 3 + 5 + \cdots + (4n-1)$ using sigma notation.

Solution:

$$\sum_{i=1}^{2n} (2i-1)$$

(b) Use Mathematical Induction to prove that for all $n \geq 1$, the above expression is equal to $(2n)^2$.

Solution: For any integer $n \geq 1$, let P_n be the statement that $1+3+5+\cdots+(4n-1) = (2n)^2$.

Base Case. The statement P_1 says that $1+3 = 2^2 = 4$, which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is,

$$1+3+5+\cdots+(4k-1) = (2k)^2.$$

It remains to show that P_{k+1} holds, that is, $1+3+5+\cdots+(4(k+1)-1) = (2(k+1))^2$, or in other words, $1+3+5+\cdots+(4k+3) = (2k+2)^2$.

$$\begin{aligned} 1+3+5+\cdots+(4k-1)+(4k+1)+(4k+3) &= (2k)^2+(4k+1)+(4k+3) \\ &= 4k^2+4k+1+4k+3 \\ &= 4k^2+8k+4 \\ &= (2k+2)^2. \end{aligned}$$

Therefore P_{k+1} holds.

Thus, by the principle of mathematical induction, for all $n \geq 1$, P_n holds. $\square$

3. Use Mathematical Induction to prove that for all $n \geq 1$,

$$n+(n+1)+(n+2)+\cdots+(2n) = \frac{3n(n+1)}{2}.$$

Solution: For any integer $n \geq 1$, let P_n be the statement that

$$n+(n+1)+(n+2)+\cdots+(2n) = \frac{3n(n+1)}{2}.$$

Base Case. The statement P_1 says that $1+2 = 3 = \frac{3(1+1)}{2} = \frac{6}{2} = 3$, which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is,

$$k+(k+1)+(k+2)+\cdots+(2k) = \frac{3k(k+1)}{2}$$

It remains to show that P_{k+1} holds, that is, $(k+1)+(k+2)+(k+3)+\cdots+(2k+2) = \frac{3(k+1)(k+2)}{2}$.

$$\begin{aligned} &(k+1)+(k+2)+(k+3)+\cdots+(2k+2) \\ &\quad = k+(k+1)+(k+2)+(k+3)+\cdots+(2k+2)-k \\ &\quad = k+(k+1)+(k+2)+\cdots+(2k)+(2k+1)+(2k+2)-k \\ &\quad = \frac{3k(k+1)}{2}+3k+3 \\ &\quad = 3(k+1)\left(\frac{k}{2}+1\right) \end{aligned}$$

$$
\begin{aligned}
&= 3(k+1)\left(\frac{k+2}{2}\right) \\
&= \frac{3(k+1)(k+2)}{2}.
\end{aligned}
$$

Therefore P_{k+1} holds.

Thus, by the principle of mathematical induction, for all $n \geq 1$, P_n holds. $\square$

4. Use Mathematical Induction to prove that for all $n \geq 1$,

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{2n}} = \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2n+1}\right).$$

Solution: For any integer $n \geq 1$, let P_n be the statement that $1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{2n}} = \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2n+1}\right)$.

Base Case. The statement P_1 says that $1 + \frac{1}{3} + \frac{1}{3^2} = \frac{9}{9} + \frac{3}{9} + \frac{1}{9} = \frac{13}{9} = \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^3\right) = \frac{3}{2}(1 - \frac{1}{27}) = \frac{3}{2} \cdot \frac{26}{27} = \frac{13}{9}$, which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is,

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{2k}} = \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2k+1}\right).$$

It remains to show that P_{k+1} holds, that is,

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{2(k+1)}} = \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2(k+1)+1}\right),$$

or in other words,

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{2k+2}} = \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2k+3}\right).$$

$$
\begin{aligned}
1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{2k+2}} &= 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{2k}} + \frac{1}{3^{2k+1}} + \frac{1}{3^{2k+2}} \\
&= \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2k+1}\right) + \frac{1}{3^{2k+1}} + \frac{1}{3^{2k+2}} \\
&= \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2k+1}\right) + \frac{1}{3 \cdot 3^{2k}} + \frac{1}{3^2 \cdot 3^{2k}} \\
&= \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2k+1}\right) + \frac{3}{2}\left(\frac{2}{3^2 \cdot 3^{2k}} + \frac{2}{3^3 \cdot 3^{2k}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2k+1} + \frac{2}{3^2 \cdot 3^{2k}} + \frac{2}{3^3 \cdot 3^{2k}}\right) \\
&= \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2k+1} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{2k+1} + \left(\frac{2}{3^2}\right)\left(\frac{1}{3}\right)^{2k+1}\right) \\
&= \frac{3}{2}\left(1 + \left(\frac{1}{3}\right)^{2k+1}\left(-1 + \frac{2}{3} + \frac{2}{3^2}\right)\right) \\
&= \frac{3}{2}\left(1 + \left(\frac{1}{3}\right)^{2k+1}\left(\frac{-1}{9}\right)\right) \\
&= \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2k+3}\right).
\end{aligned}
$$

Therefore P_{k+1} holds.

Thus, by the principle of mathematical induction, for all $n \geq 1$, P_n holds. □

5. Use Mathematical Induction to prove that for all $n \geq 1$, $3^n > n^2$.

Solution: For any integer $n \geq 1$, let P_n be the statement that $3^n > n^2$.

Base Case. The statement P_1 says that $3^1 = 3 > 1^2 = 1$, which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is,

$$3^k > k^2.$$

It remains to show that P_{k+1} holds, that is, $3^{k+1} > (k+1)^2 = k^2 + 2k + 1$.

$$
\begin{aligned}
3^{k+1} &= 3 \cdot 3^k \\
&= 3^k + 3^k + 3^k \\
&> k^2 + 2^k + 1 \\
&\geq k^2 + 2k + 1 \\
&= (k+1)^2.
\end{aligned}
$$

Therefore P_{k+1} holds.

Thus, by the principle of mathematical induction, for all $n \geq 1$, P_n holds. □

6. Consider the sequence of real numbers defined by the relations $x_0 = 0.5$ and for all $n \geq 1$, $x_n = 0.5x_{n-1}(1 - x_{n-1})$. Prove by induction that for all $n \geq 0$, $x_n \in (0, 1)$.

Solution: For any $n \geq 0$, let P_n be the statement that $x_n \in (0, 1)$.

Base Case. The statement P_0 says that $x_0 = 0.5 \in (0, 1)$, which is true.

Inductive Step. Fix $k \geq 0$, and suppose that P_k holds, that is, $x_k \in (0, 1)$. Then $x_k < 1$, $x_k > 0$, $-x_k < 0$, and $-x_k > -1$.

It remains to show that P_{k+1} holds, that is, that $x_{k+1} \in (0, 1)$.

$$\begin{aligned} x_{k+1} &= 0.5x_k(1 - x_k) \\ &< 0.5(1)(1) = 0.5, \text{ and} \\ x_{k+1} &= 0.5x_k(1 - x_k) \\ &> 0.5(0)(1 + -1) = 0. \end{aligned}$$

Therefore $x_{k+1} \in (0, 0.5)$, and so P_{k+1} holds.

Thus by the principle of mathematical induction, for all $n \geq 1$, P_n holds.

7. (a) Express the sum $\sum_{k=1}^{m}(2 + 3k)^2$ in terms of three simpler sums in sigma notation by expanding. Do not calculate the value.

Solution:

$$\begin{aligned} \sum_{k=1}^{m}(2 + 3k)^2 &= \sum_{k=1}^{m} 4 + 12k + 9k^2 \\ &= \sum_{k=1}^{m} 4 + \sum_{k=1}^{m} 12k + \sum_{k=1}^{m} 9k^2 \\ &= 4\sum_{k=1}^{m} 1 + 12\sum_{k=1}^{m} k + 9\sum_{k=1}^{m} k^2. \end{aligned}$$

(b) Find the value of the sum

$$\sum_{p=1}^{100}(2 - 10p + 3p^2).$$

HINT: Make use of the formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \text{and} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution:

$$\sum_{p=1}^{100}(2 - 10p + 3p^2) = \sum_{p=1}^{100} 2 - \sum_{p=1}^{100} 10p + \sum_{p=1}^{100} 3p^2$$

$$
\begin{aligned}
&= 2\sum_{p=1}^{100} 1 - 10\sum_{p=1}^{100} p + 3\sum_{p=1}^{100} p^2 \\
&= 2(100) - 10\frac{100(100+1)}{2} + 3\frac{100(100+1)(200+1)}{6} \\
&= 200 - 500(100+1) + 50(100+1)(200+1) \\
&= 200 - 50500 + 1015050 \\
&= 964750.
\end{aligned}
$$

(c) Rewrite the sum

$$\sum_{r=12}^{122} \frac{r-6}{r+9}$$

using an index whose initial and terminal values are 1 and 111 (HINT: use a change of variables).

Solution: Let $i = r - 11$. Then, $r - 6 = i + 5$ and $r + 9 = i + 20$, and so,

$$\sum_{r=12}^{122} \frac{r-6}{r+9} = \sum_{i=1}^{111} \frac{i+5}{i+20}.$$