This assignment is **optional** and does not need to be handed in. Attempt all questions, write out nicely written solutions (showing all your work), and the solutions will be posted on Fri, Mar 10, 2017, at which point you can mark your own work. If you have any questions regarding differences between what you wrote and what the solution key says, please contact your professor. **At least one question from this assignment will be found on Quiz 3.**

Note: Throughout this assignment, the length of a vector $\mathbf{v}$ is denoted by $||\mathbf{v}||$ (while the book uses $|\mathbf{v}|$).

1. Let $\mathbf{u} = [1, 1, 1]$, $\mathbf{v} = [-1, 2, 5]$, $\mathbf{w} = [0, 1, 1]$. Calculate each of the following:

   (a) $(2\mathbf{u} + \mathbf{v}) \bullet (\mathbf{v} - 3\mathbf{w})$

   Solution:

$$\begin{aligned}(2\mathbf{u} + \mathbf{v}) \bullet (\mathbf{v} - 3\mathbf{w}) &= (2[1,1,1] + [-1,2,5]) \bullet ([-1,2,5] - 3[0,1,1]) \\ &= ([2,2,2] + [-1,2,5]) \bullet ([-1,2,5] - [0,3,3]) \\ &= [1,4,7] \bullet [-1,-1,2] \\ &= -1 - 4 + 14 = 9.\end{aligned}$$

   (b) $||\mathbf{u}|| - 2||\mathbf{v}|| + ||(-3)\mathbf{w}||$

   Solution:

$$\begin{aligned}||\mathbf{u}|| - 2||\mathbf{v}|| + ||(-3)\mathbf{w}|| &= ||[1,1,1]|| - 2||[-1,2,5]|| + ||(-3)[0,1,1]|| \\ &= \sqrt{1^2 + 1^2 + 1^2} - 2\sqrt{1^2 + 2^2 + 5^2} + ||[0,-3,-3]|| \\ &= \sqrt{3} - 2\sqrt{30} + \sqrt{3^2 + 3^2} \\ &= \sqrt{3} - 2\sqrt{30} + \sqrt{18}.\end{aligned}$$

2. Prove the associative rule for addition of vectors in $\mathbb{R}^3$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

   in the following two different ways:

   (a) by writing each of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in terms of their coordinates and simplifying both sides algebraically in coordinate form

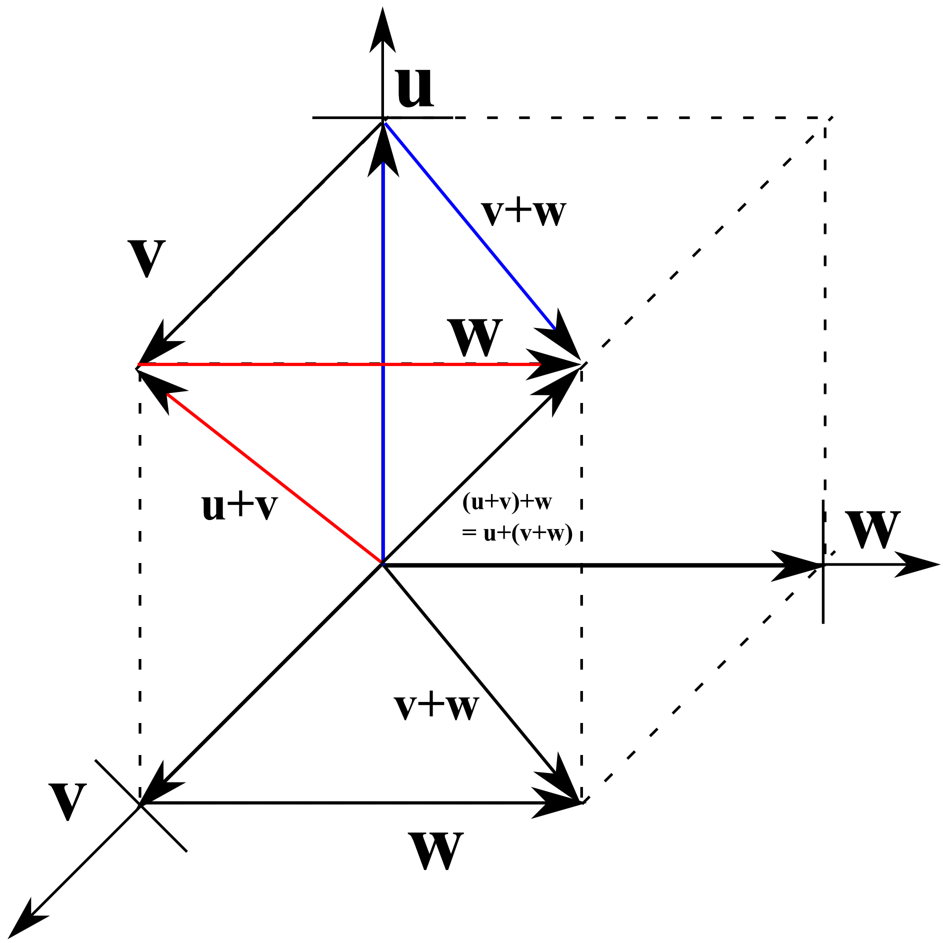
   Solution: Let $\mathbf{u} = (a, b, c)$, $\mathbf{v} = (d, e, f)$, $\mathbf{w} = (x, y, z)$. Then

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= ((a,b,c) + (d,e,f)) + (x,y,z) \\ &= (a+d, b+e, c+f) + (x,y,z) \\ &= ((a+d)+x, (b+e)+y, (c+f)+z) \\ &= (a+(d+x), b+(e+y), c+(f+z))\end{aligned}$$

$$
\begin{aligned}
&= (a,b,c) + (d+x, e+y, f+z) \\
&= (a,b,c) + ((d,e,f) + (x,y,z)) \\
&= \mathbf{u} + (\mathbf{v} + \mathbf{w}).
\end{aligned}
$$

(b) by a geometric argument using arrow representations for **u**, **v**, **w**

Solution:



3. Find the points where the plane $3x + 2y + 5z = 30$ meets each of the x, y and z axes in $\mathbb{R}^3$. Use these "intercepts" to provide a neat sketch of the plane.

Solution: Here are the formulae for the axes:

x-axis: $\mathbf{x} = (t, 0, 0),\ t \in \mathbb{R}$

y-axis: $\mathbf{x} = (0, t, 0),\ t \in \mathbb{R}$

z-axis: $\mathbf{x} = (0, 0, t),\ t \in \mathbb{R}$

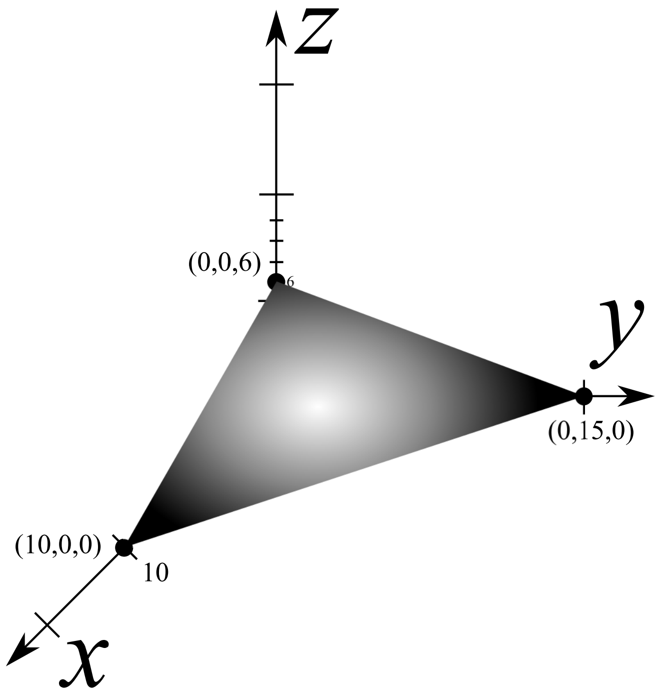
So to find the intersection of this plane and the x-axis, just plug in $y = z = 0$:

$$3x = 30 \implies x = 10 \implies (10, 0, 0)$$

Similarly,

$$2y = 30 \implies y = 15 \implies (0, 15, 0)$$
$$5z = 30 \implies z = 6 \implies (0, 0, 6)$$

Therefore we have the following plane:



4. (a) Find an equation for the line through points $(1, 3)$ and $(5, 4)$ in parametric form.

Solution: The vector $\mathbf{v} = (5, 4) - (1, 3) = (4, 1)$ is along the line. Therefore the line in point-parallel form (that is, the vector equation for the line) is:

$$\mathbf{x} = (1, 3) + t(4, 1), \ t \in \mathbb{R}$$

which in parametric form becomes

$$x = 1 + 4t, \qquad y = 3 + t, \ t \in \mathbb{R}.$$

(b) Find an equation for the line through points $(1, 2, 3)$ and $(5, 5, 0)$ in parametric form.

Solution: The vector $\mathbf{v} = (5, 5, 0) - (1, 2, 3) = (4, 3, -3)$ is along the line. Therefore the line in point-parallel form (that is, the vector equation for the line) is:

$$\mathbf{x} = (5, 5, 0) + t(4, 3, -3), \ t \in \mathbb{R}$$

which in parametric form becomes

$$x = 5 + 4t, \qquad y = 5 + 3t, \qquad z = -3t, \ t \in \mathbb{R}.$$

5. Find all unit vectors in xyz-space that are perpendicular to the x-axis and the line through the points $(1,2,3)$ and $(3,-2,1)$.

Solution: We are looking for vectors $\mathbf{x}$ that are perpendicular to the x-axis (that is, perpendicular to $(1,0,0)$), and for $P=(1,2,3)$, $Q=(3,-2,1)$, perpendicular to the direction vector $\mathbf{v}=\overrightarrow{PQ}=Q-P=(3,-2,1)-(1,2,3)=(2,-4,-2)$.

So we need $\mathbf{x}$ to be any scalar multiple of

$$(1,0,0)\times(2,-4,-2)=(0,2,-4),$$

that is, $\mathbf{x}=k(0,2,-4)$.

Finally, we need it to be a unit vector. Therefore

$$1=||\mathbf{x}||=|k|\ ||(0,2,-4)||=|k|\sqrt{0^2+2^2+(-4)^2}=|k|\sqrt{20}$$

$$|k|=\frac{1}{\sqrt{20}}=\frac{\sqrt{20}}{20}=\frac{\sqrt{5}}{10}\implies k=\frac{\pm\sqrt{5}}{10}$$

Therefore we have two solutions:

$$\mathbf{x}=\frac{\pm\sqrt{5}}{10}(0,2,-4).$$

6. Find the line through $(3,1,-2)$ that intersects and is perpendicular to the line $x=-1+t$, $y=-2+t$, $z=-1+t$ with $t\in\mathbb{R}$.

Solution: To find the equation of the line L we need a point and a direction vector. We have a point $(3,1,-2)$; we have to find a direction vector $\mathbf{v}$. The line L is perpendicular to the line L_1, which is passing through the point $(-1,-2,-1)$ and having a direction vector $\langle 1,1,1\rangle$, so $\mathbf{v}$ is perpendicular to $\langle 1,1,1\rangle$. The line L passes through $(3,1,-2)$ and intersects the line L_1 at a point P, so P is on L and also on L_1. So P satisfies $x=-1+t$, $y=-2+t$, $z=-1+t$ with $t\in\mathbb{R}$; we need to find t corresponding to this point.

The vector starting at $(3,1,-2)$ and finishing at P is along the line L, so it is a direction vector for L: $\mathbf{v}=\langle 3-(-1+t),1-(-2+t),-2-(-1+t)\rangle=\langle 4-t,3-t,-1-t\rangle$. As $\mathbf{v}$ is perpendicular to $\langle 1,1,1\rangle$, we have

$$\langle 4-t,3-t,-1-t\rangle\cdot\langle 1,1,1\rangle=4-t+3-t-1-t=6-3t=0$$

so $t=2$. Therefore, $\mathbf{v}=\langle 4-2,3-2,-1-2\rangle=\langle 2,1,-3\rangle$. Using $(3,1,-2)$ and $\mathbf{v}=\langle 2,1,-3\rangle$ the parametric equations for L are

$$\begin{aligned}x=&3+2t,\\ y=&1+1t,\\ z=&-2-3t,\end{aligned}$$

and the symmetric equations are

$$\frac{x-3}{2} = y-1 = \frac{-z-2}{3}.$$

7. Find an equation for the plane containing the two lines of equations

$$\langle x, y, z\rangle = \langle 0, 1, 2\rangle + t_1\langle 2, 3, -1\rangle, \quad t_1 \in \mathbb{R}$$

and

$$\langle x, y, z\rangle = \langle 2, -1, 0\rangle + t_2\langle 4, 6, -2\rangle, \quad t_2 \in \mathbb{R}.$$

Solution: To find the equation of a plane we need a point in the plane and its normal vector. We have 2 lines: L_1 passing through the point $(0, 1, 2)$ and having as direction vector $\mathbf{v_1} = \langle 2, 3, -1\rangle$ and L_2 passing through the point $(2, -1, 0)$ and having as direction vector $\mathbf{v_2} = \langle 4, 6, -2\rangle$. We note that L_1 and L_2 are parallel as their direction vectors are in the same direction, $2\mathbf{v_1} = \mathbf{v_2}$.

If we have 2 (non-parallel) vectors belonging to the plane by using their cross product we will obtain the vector normal to the plane. We have already one vector $\mathbf{v_1}$ (or $\mathbf{v_2}$); as a second vector we consider the vector starting from $(0, 1, 2)$ (point of L_1) and finishing at $(2, -1, 0)$ (point of L_2). This vector $\langle 2-0, -1-1, 0-2\rangle = \langle 2, -2, -2\rangle$ belongs to the plane as the 2 points belong to the plane. To find the normal vector to the plane, compute the cross product of $\langle 2, -2, -2\rangle$ and $\mathbf{v_1}$ (or $\mathbf{v_2}$):

$$\begin{aligned}\langle 2, -2, -2\rangle \times \langle 2, 3, -1\rangle &= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 2 & -2 & -2 \\ 2 & 3 & -1 \end{vmatrix} = \begin{vmatrix} -2 & -2 \\ 3 & -1 \end{vmatrix}\vec{\mathbf{i}} - \begin{vmatrix} 2 & -2 \\ 2 & -1 \end{vmatrix}\vec{\mathbf{j}} + \begin{vmatrix} 2 & -2 \\ 2 & 3 \end{vmatrix}\vec{\mathbf{k}} \\ &= 8\vec{\mathbf{i}} - 2\vec{\mathbf{j}} + 10\vec{\mathbf{k}}\end{aligned}$$

For the equation of the plane, we use the point $(2, -1, 0)$ and the normal vector $\langle 8, -2, 10\rangle$ and obtain

$$8(x-2) - 2(y+1) + 10z = 0.$$

8. Show that the lines $x - 3 = 4t_1$, $y - 4 = t_1$, $z - 1 = 0$ and $x + 1 = 12t_2$, $y - 7 = 6t_2$, $z - 5 = 3t_2$ with $t_1, t_2 \in \mathbb{R}$ intersect, and find the point of intersection.

Solution: The line L_1 has a direction vector $\langle 4, 1, 0\rangle$ and the line L_2 has a direction vector $\langle 12, 6, 3\rangle$. The 2 direction vectors are not parallel (it is not possible to express one vector as a multiple of the other one).

An intersection point must satisfy both equations of L_1 and L_2. From the equations of L_1, we know that all points on L_1 must have $z = 1$. Using $z = 1$ and the equation for the $z-$coordinates of L_2 points $z - 5 = 3t_2$, we can characterize in a unique way the value for $t_2 : z = 1 = 5 + 3t_2$ then $t_2 = -4/3$. Using $t_2 = -4/3$ in L_2 equations, we obtain the following point $(-1 + 12 \times (-4/3), 7 + 6 \times (-4/3), 1) = (-17, -1, 1)$ which belongs to L_2. Check if this point $(-17, -1, 1)$ belongs to L_1 (satisfies the L_1 equations):

$$\begin{aligned} -17 &= 3 + 4t_1 && \Rightarrow t_1 = -5 \\ -1 &= 4 + t_1 && \Rightarrow t_1 = -5 \\ 1 &= 1 \end{aligned}$$

Using $t_1 = -5$ in L_1 equations, we find $(-17, -1, 1)$. Therefore, $(-17, -1, 1)$ is a point of both lines L_1 and L_2. Since the two lines are not parallel, the intersection point is $(-17, -1, 1)$.