This assignment is **optional** and does not need to be handed in. Attempt all questions, write out nicely written solutions (showing all your work), and the solutions will be posted on Fri, Apr 7, 2017, at which point you can mark your own work. If you have any questions regarding differences between what you wrote and what the solution key says, please contact your professor. **At least one question from this assignment will be found on Quiz 5.**

1. A linear system of equations has 4 variables, a, b, c, and d. The RREF of the augmented matrix for this system is
$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$
Find three different solutions for this system.

2. Find all values for a and b such that $\begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -a & a \\ b & -b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

3. Write the vector $(22, -13, -1)$ as a linear combination of $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (0, 1, 2)$, $\mathbf{v}_3 = (5, -3, -1)$. Use any method and show all your work.

4. Consider the following augmented matrix for a linear system of equations:
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$$
Find **all values** for a that will result in this system having **infinitely many solutions**. Justify your answer.

5. What can be said about the number of solutions to a system of equations given that the RREF of the coefficient matrix contains a zero row? Explain and justify your answer as appropriate.

6. Prove using mathematical induction that for any $n \geq 2$, and collection of n $m \times m$ matrices $A_1, A_2, \ldots, A_n$,
$$\det(A_1 A_2 \cdots A_n) = \det(A_1)\det(A_2)\cdots\det(A_n).$$

7. Is it true that for any two matrices A and B,
$$\det(A + B) = \det(A) + \det(B)?$$
If so, prove it. If not, find a counter example.

8. Solve the following system using Cramer's Rule:
$$\begin{array}{rcrcrcr} x_1 & & & + & 3x_3 & = & -1 \\ & - & x_2 & + & 2x_3 & = & -9 \\ 2x_1 & + & x_2 & & & = & 15 \end{array}$$

9. Prove the following property: for all $a, b, c \in \mathbb{R}$, $a \neq 0$, $b \neq 0$, $c \neq 0$,

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

10. (a) Let $c \in \mathbb{R}$. Prove using mathematical induction that for any $n \geq 1$ and any $n \times n$ matrix A, $|cA| = c^n|A|$.

    (b) A square matrix is called **skew-symmetric** if $A^T = -A$. Use part (a) and a property of determinants when taking transposes to show that every skew-symmetric 1001×1001 matrix has determinant 0.

11. Find the inverse of the matrix or explain why the inverse does not exist.

    (a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

    (b) $B = \begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$

12. Find all values of c, if any, for which the matrix $A = \begin{bmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{bmatrix}$ is invertible.

13. Show that if A is invertible, then $\det(A^{-1}) = \det(A)^{-1}$. Deduce a formula for the determinant of $4A^{-1}$, when A is an invertible $n \times n$-matrix.

14. Writing the system

$$\begin{array}{llll} x_1 & & +x_3 & = 4 \\ 2x_1 & +3x_2 & +5x_3 & = -3 \\ x_1 & & +2x_3 & = 0 \end{array}$$

    as $A\mathbf{x} = \mathbf{b}$,

    (a) find the inverse matrix A^{-1};

    (b) find the solution to the system $A^T\mathbf{x} = \mathbf{b}$ by using (a).