This assignment is **optional** and does not need to be handed in. Attempt all questions, write out nicely written solutions (showing all your work), and the solutions will be posted on Fri, Apr 7, 2017, at which point you can mark your own work. If you have any questions regarding differences between what you wrote and what the solution key says, please contact your professor. **At least one question from this assignment will be found on Quiz 5.**

1. A linear system of equations has 4 variables, a, b, c, and d. The RREF of the augmented matrix for this system is

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Find three different solutions for this system.

Solution: There are infinitely many valid solutions. Any three solutions of the form

$$a = -1 - t + 2s, \quad b = s, \quad c = 2 - 3t, \quad d = t, \quad s, t \in \mathbb{R}$$

would suffice. For instance,

$$\textbf{Solution \#1}: a = -1, \quad b = 0, \quad c = 2, \quad d = 0 \quad (s = t = 0)$$

$$\textbf{Solution \#2}: a = 1, \quad b = 1, \quad c = 2, \quad d = 0 \quad (s = 1, t = 0)$$

$$\textbf{Solution \#3}: a = -2, \quad b = 0, \quad c = -1, \quad d = 1 \quad (s = 0, t = 1)$$

2. Find all values for a and b such that $\begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -a & a \\ b & -b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} a + 2b & -a - 2b \\ 2a + 4b & -2a - 4b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$a + 2b = 0, \quad -a - 2b = 0, \quad 2a + 4b = 0, \quad -2a - 4b = 0.$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & -2 & 0 \\ 2 & 4 & 0 \\ -2 & -4 & 0 \end{array}\right] \quad \begin{array}{c} R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 - 2R_1 \\ R_4 \leftarrow R_4 + 2R_1 \end{array} \quad \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$a = -2t, \quad b = t, \quad t \in \mathbb{R}$$

3. Write the vector $(22, -13, -1)$ as a linear combination of $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (0, 1, 2)$, $\mathbf{v}_3 = (5, -3, -1)$. Use any method and show all your work.

Solution:

$$a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\1\\2\end{bmatrix}+c\begin{bmatrix}5\\-3\\-1\end{bmatrix}=\begin{bmatrix}22\\-13\\-1\end{bmatrix}$$

$$\begin{bmatrix}a\\0\\0\end{bmatrix}+\begin{bmatrix}0\\b\\2b\end{bmatrix}+\begin{bmatrix}5c\\-3c\\-c\end{bmatrix}=\begin{bmatrix}22\\-13\\-1\end{bmatrix}$$

$$\begin{bmatrix}a+5c\\b-3c\\2b-c\end{bmatrix}=\begin{bmatrix}22\\-13\\-1\end{bmatrix}$$

$$a+5c=22,\qquad b-3c=-13,\qquad 2b-c=-1.$$

$$\left[\begin{array}{ccc|c}1&0&5&22\\0&1&-3&-13\\0&2&-1&-1\end{array}\right]$$

$$R_3 \leftarrow R_3-2R_2$$

$$\left[\begin{array}{ccc|c}1&0&5&22\\0&1&-3&-13\\0&0&5&25\end{array}\right]$$

$$R_3 \leftarrow \frac{1}{5}R_3$$

$$\left[\begin{array}{ccc|c}1&0&5&22\\0&1&-3&-13\\0&0&1&5\end{array}\right]$$

$$R_1 \leftarrow R_1-5R_3$$

$$R_2 \leftarrow R_2+3R_3$$

$$\left[\begin{array}{ccc|c}1&0&0&-3\\0&1&0&2\\0&0&1&5\end{array}\right]$$

$$a=-3$$

$$b=2$$

$$c=5$$

4. Consider the following augmented matrix for a linear system of equations:

$$\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&a^2-4&a-2\end{array}\right]$$

Find **all values** for a that will result in this system having **infinitely many solutions**. Justify your answer.

Solution: To have infinitely many solutions, the last row must be a zero row, forcing $a^2-4=a-2=0$. The only value for a that makes this true is $a=2$.

5. What can be said about the number of solutions to a system of equations given that the RREF of the coefficient matrix contains a zero row? Explain and justify your answer as appropriate.

Solution: Nothing. The number of solutions is completely unknown. For instance, consider the system

$$x+y=4,\quad x-y=0,\quad 2x+2y=8.$$

The RREF of the coefficient matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}$ has a zero row, but there is exactly one solution. Similarly, the system

$$x + y = 4, \quad x - y = 0, \quad x + y = 2$$

is such that the RREF of the coefficient matrix has a zero row, but there are no solutions (how can $x + y = 4$ and 2 at the same time). Finally,

$$x + y = 4, \quad 2x + 2y = 8$$

has coefficient matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, and it's RREF has a zero row, which produces infinitely many solutions. The size could be anything, the number of solutions could be anything. Really, the most you can say is that one of the rows of the coefficient matrix was a linear combination of the other rows, but this is something about the coefficient matrix, not really about the system itself.

6. Prove using mathematical induction that for any $n \geq 2$, and collection of n $m \times m$ matrices $A_1, A_2, \ldots, A_n$,

$$\det(A_1 A_2 \cdots A_n) = \det(A_1)\det(A_2)\cdots\det(A_n).$$

Solution: Fix $m \geq 1$. For all $n \geq 2$, let P_n denote the statement that for any collection of n $m \times m$ matrices $A_1, A_2, \ldots, A_n$,

$$\det(A_1 A_2 \cdots A_n) = \det(A_1)\det(A_2)\cdots\det(A_n).$$

Base Case. The statement P_2 says that for any collection of 2 $m \times m$ matrices A, B,

$$\det(AB) = \det(A)\det(B).$$

This is true by Theorem 7.6 in the text.

Inductive Step. Fix $k \geq 2$ and suppose that P_k holds, that is, for any collection of k $m \times m$ matrices $A_1, A_2, \ldots, A_k$,

$$\det(A_1 A_2 \cdots A_k) = \det(A_1)\det(A_2)\cdots\det(A_k).$$

It remains to show that P_{k+1} holds, that is, for any collection of $k+1$ $m \times m$ matrices $A_1, A_2, \ldots, A_{k+1}$,

$$\det(A_1 A_2 \cdots A_{k+1}) = \det(A_1)\det(A_2)\cdots\det(A_{k+1}).$$

Let $A_1, A_2, \ldots, A_{k+1}$ be $m \times m$ matrices. Then

$$\begin{aligned}
\det(A_1 A_2 \cdots A_{k+1}) &= \det(\ (A_1 A_2 \cdots A_k)\ A_{k+1}\ ) \\
&= \det(A_1 A_2 \cdots A_k)\ \det(A_{k+1}) && \text{(base case)} \\
&= \det(A_1)\det(A_2)\cdots\det(A_k)\det(A_{k+1}) && \text{(by } P_k\text{)}.
\end{aligned}$$

Therefore P_{k+1} holds. Thus by PMI, for all $n \geq 2$, P_n holds.

7. Is it true that for any two matrices A and B,

$$\det(A+B) = \det(A) + \det(B)?$$

If so, prove it. If not, find a counter example.

Solution: No. For instance,

$$\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, \qquad \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0,$$

but

$$\left| \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}.$$

8. Solve the following system using Cramer's Rule:

$$\begin{array}{rcrcrcr} x_1 & & & + & 3x_3 & = & -1 \\ & - & x_2 & + & 2x_3 & = & -9 \\ 2x_1 & + & x_2 & & & = & 15 \end{array}$$

Solution:

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -9 \\ 15 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -1 & 0 & 3 \\ -9 & -1 & 2 \\ 15 & 1 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & -1 & 3 \\ 0 & -9 & 2 \\ 2 & 15 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -9 \\ 2 & 1 & 15 \end{bmatrix}$$

Then $|A| = 4$, $|A_1| = 20$, $|A_2| = 20$, and $|A_3| = -8$. Therefore, by Cramer's rule:

$$x_1 = \frac{|A_1|}{|A|} = \frac{20}{4} = 5$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{20}{4} = 5$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{-8}{4} = -2.$$

9. Prove the following property: for all $a, b, c \in \mathbb{R}$, $a \neq 0$, $b \neq 0$, $c \neq 0$,

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Solution:

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} R_2 \leftarrow R_2 - R_1$$

$$= \begin{vmatrix} 1+a & 1 & 1 \\ -a & b & 0 \\ 1 & 1 & 1+c \end{vmatrix} R_3 \leftarrow R_3 - (1+c)R_1$$

$$= \begin{vmatrix} 1+a & 1 & 1 \\ -a & b & 0 \\ 1-(1+a)(1+c) & -c & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a & 1 & 1 \\ -a & b & 0 \\ -a-c-ac & -c & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -a & b \\ -a-c-ac & -c \end{vmatrix}$$

$$= ac - b(-a-c-ac) = ac + ba + bc + abc = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

10. (a) Let $c \in \mathbb{R}$. Prove using mathematical induction that for any $n \geq 1$ and any $n \times n$ matrix A, $|cA| = c^n|A|$.

Solution: Fix $c \in \mathbb{R}$. For all $n \geq 1$, let P_n denote the statement that for any $n \times n$ matrix A, $|cA| = c^n|A|$.

Base Case. The statement P_1 says that for any 1×1 matrix $A = [a_{1,1}]$, $|cA| = c|A|$.

$$|cA| = |[ca_{1,1}]| = ca_{1,1} = c|[a_{1,1}]| = c|A|.$$

Therefore P_1 holds.

Inductive Step. Fix $k \geq 1$ and assume that P_k holds, that is, for any $k \times k$ matrix A, $|cA| = c^k|A|$. It remains to show that P_{k+1} holds, that is, for any $(k+1) \times (k+1)$ matrix A, $|cA| = c^{k+1}|A|$.

First some notation: let $A_{i,j}$ denote the matrix formed from A by removing row i and column j. Then expanding across the first row we have:

$$\begin{aligned} |cA| &= \sum_{j=1}^{k+1} ca_{1,j}C_{1,j} && \text{Recall } C_i, j \text{ is the cofactor at } i, j \\ &= \sum_{j=1}^{k+1} ca_{1,j}(-1)^{1+j}|(cA)_{1,j}| \\ &= \sum_{j=1}^{k+1} ca_{1,j}(-1)^{1+j}c^k|A_{1,j}| && \text{By } P_k \text{ since } (cA)_{1,j} \text{ is a } k \times k \text{ matrix} \end{aligned}$$

$$= c^{k+1}\sum_{j=1}^{k+1} a_{1,j}(-1)^{1+j}|A_{1,j}|$$
$$= c^{k+1}|A|.$$

Therefore P_{k+1} holds, and thus by PMI, for all $n \geq 1$, P_n holds.

(b) A square matrix is called **skew-symmetric** if $A^T = -A$. Use part (a) and a property of determinants when taking transposes to show that every skew-symmetric 1001×1001 matrix has determinant 0.

Solution: Let A be a skew-symmetric matrix. Then $A^T = -A$. Taking the determinant of both sides, we get

$$\begin{aligned} |A^T| &= |A| \\ |-A| &= |(-1)A| \\ &= (-1)^{1001}|A| \\ &= -|A|. \end{aligned}$$

Thus $|A| = -|A|$, and so $2|A| = 0$, thus $|A| = 0$.

11. Find the inverse of the matrix or explain why the inverse does not exist.

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Solution: Compute $\det(A)$ by expanding along the bottom row:

$$\det(A) = \begin{vmatrix} 2 & 3 \\ 5 & 3 \end{vmatrix} + 8\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 6 - 15 + 40 - 32 = -1,$$

so A is invertible. We compute the inverse using the adjoint. The matrix of cofactors is

$$C = \begin{bmatrix} 40 & -13 & -5 \\ -16 & 5 & 2 \\ -9 & 3 & 1 \end{bmatrix},$$

so

$$A^{-1} = \frac{1}{\det(A)}C^T = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}.$$

(b) $B = \begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$

Solution: We have

$$\det(B) = \begin{vmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix} \underset{R_3 \leftarrow R_3 - 4R_1}{\overset{R_2 \leftarrow R_2 + 2R_1}{=}} \begin{vmatrix} -1 & 3 & -4 \\ 0 & 10 & -7 \\ 0 & -10 & 7 \end{vmatrix}$$

and thus $\det(B) = 0$ since the third row is a multiple of the second (and thus adding, for instance, row 2 to row 3, would lead to a row of zeros). So B is not invertible.

12. Find all values of c, if any, for which the matrix $A = \begin{bmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{bmatrix}$ is invertible.

Solution: A is invertible iff its determinant is nonzero. Expanding along, say, the first row, we find

$$\det(A) = c \begin{vmatrix} c & 1 \\ 1 & c \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & c \end{vmatrix} = c(c^2 - 1) - c = c(c^2 - 2).$$

Therefore, A is invertible iff $c \neq 0$ and $c^2 - 2 \neq 0$, i.e., $c \neq 0$, $c \neq \sqrt{2}$ and $c \neq -\sqrt{2}$.

13. Show that if A is invertible, then $\det(A^{-1}) = \det(A)^{-1}$. Deduce a formula for the determinant of $4A^{-1}$, when A is an invertible $n \times n$-matrix.

Solution: If A is invertible, then A^{-1} exists such that $AA^{-1} = I$. Take the determinant of both sides and use the fact that $\det(AB) = \det(A)\det(B)$:

$$\det(AA^{-1}) = \det(I) \Leftrightarrow \det(A)\det(A^{-1}) = \det(I) \Leftrightarrow \det(A)\det(A^{-1}) = 1$$

and thus, dividing both sides of the latter equality by $\det(A)$,

$$\det(A^{-1}) = \det(A)^{-1}.$$

Recall that if all entries in a row of A are multiplied by some constant k, then the determinant of A is multiplied by k. So, if A is $n \times n$, if all entries of A are multiplied by k, then the determinant of A is multiplied by k^n. Therefore,

$$\det(4A^{-1}) = 4^n \det(A^{-1}) = \frac{4^n}{\det(A)}.$$

14. Writing the system

$$\begin{array}{llll} x_1 & & +x_3 & = 4 \\ 2x_1 & +3x_2 & +5x_3 & = -3 \\ x_1 & & +2x_3 & = 0 \end{array}$$

as $A\mathbf{x} = \mathbf{b}$,

(a) find the inverse matrix A^{-1};

Solution: The matrix takes the form

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}.$$

We compute the determinant by expanding along the second column:

$$\det(A) = 3 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 3,$$

so the matrix A is invertible. We invert it, for example, by row reduction. We have

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \underset{R_3 \leftarrow R_3 - R_1}{\xrightarrow{R_2 \leftarrow R_2 - 2R_1}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\underset{R_2 \leftarrow R_2 - 3R_1}{\xrightarrow{R_1 \leftarrow R_1 - R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 3 & 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2/3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] = [I|A^{-1}].$$

(b) find the solution to the system $A^T\mathbf{x} = \mathbf{b}$ by using (a).

Solution: Since $\det(A) = \det(A^T)$, the system $A^T\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = (A^T)^{-1}\mathbf{b}$. Recall that $(A^T)^{-1} = (A^{-1})^T$. [Indeed, suppose A is invertible. Then $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$, thus the inverse of A^T is $(A^{-1})^T$.]
Thus

$$\mathbf{x} = (A^T)^{-1}\mathbf{b} = (A^{-1})^T\mathbf{b} = \begin{bmatrix} 2 & \frac{1}{3} & -1 \\ 0 & \frac{1}{3} & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ -1 \end{bmatrix}.$$