UNIVERSITY OF MANITOBA
DEPARTMENT OF MATHEMATICS
MATH 1210 Techniques of Classical and Linear Algebra
MIDTERM EXAM
2017-03-06 5:30-7:00pm

FAMILY / LAST NAME: (Print in ink) ______________________

FIRST NAME: (Print in ink) Solutions

STUDENT NUMBER: ______________________

SIGNATURE: (in ink) ______________________
(I understand that cheating is a serious offense)

INSTRUCTIONS TO STUDENTS:

This is a 90 minute exam.

Fill in all the information above.

No *calculators, texts, notes, cell phones, pagers, translators or other electronics are permitted. No outside paper is permitted.*

This exam has a title page and 9 other pages, 2 of which can be used for rough work. You may remove the scrap pages if you like, but be careful not to loosen the staple. Please check that you have all pages.

Important: Any work you want marked must be written on the front of a page. If you need more room, write your answers on the scrap paper. Anything written on the back of pages will not be marked (so feel free to use these for rough work).

There are 7 questions for a total of 80 marks.
Show all your work clearly and justify your answers using complete sentences (unless it is explicitly stated that you do not have to do that). **Unjustified answers may receive LITTLE or NO CREDIT.**

If a question calls for a specific method, no credit will be given for other methods.

[15] **1.** Prove by induction that for all $n \geq 1$, $\sum_{i=1}^{2n} 3i - 2 = n(6n-1)$. Show all your work, use complete sentences, and use the style done in class.

For all $n \geq 1$, let P_n denote the statement $\sum_{i=1}^{2n} 3i-2 = n(6n-1)$.

Base Case. The statement P_1 says

$$1+4 = \sum_{i=1}^{2(1)} 3i-2 = 1(6(1)-1) = 5,$$

which is true.

Inductive Step. Fix $k \geq 1$, and assume P_k holds, that is, $\sum_{i=1}^{2k} 3i-2 = k(6k-1)$. It remains to show P_{k+1} holds, that is, $\sum_{i=1}^{2(k+1)} 3i-2 = (k+1)(6(k+1)-1)$.

$$\sum_{i=1}^{2(k+1)} 3i-2 = \sum_{i=1}^{2k+2} 3i-2$$
$$= \sum_{i=1}^{2k} 3i-2 + 3(2k+1)-2 + 3(2k+2)-2$$
$$= k(6k-1) + 6k+3 + 6k+6 - 4$$
$$= (k+1)(6k-1) - (6k-1) + 12k+5$$
$$= (k+1)(6k-1) + 6k+6$$
$$= (k+1)(6k-1+6)$$
$$= (k+1)(6k+5) = (k+1)(6(k+1)-1).$$

Thus P_{k+1} holds.

Therefore, by PMI, for all $n \geq 1$, P_n holds.

[3] **2.** Write the following sum using sigma notation starting at 1 (do not evaluate):

$$\frac{-4}{5}+\frac{2}{5}-\frac{4}{15}+\frac{1}{5}-\frac{4}{25}+\frac{2}{15}$$

$$-\frac{4}{5}+\frac{4}{10}-\frac{4}{15}+\frac{4}{20}-\frac{4}{25}+\frac{4}{30}$$

$$\sum_{i=1}^{6} \frac{(-1)^i \cdot 4}{5i}$$

[8] **3.** Let z_1 and z_2 be 2 complex numbers. Prove that

$$\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$$

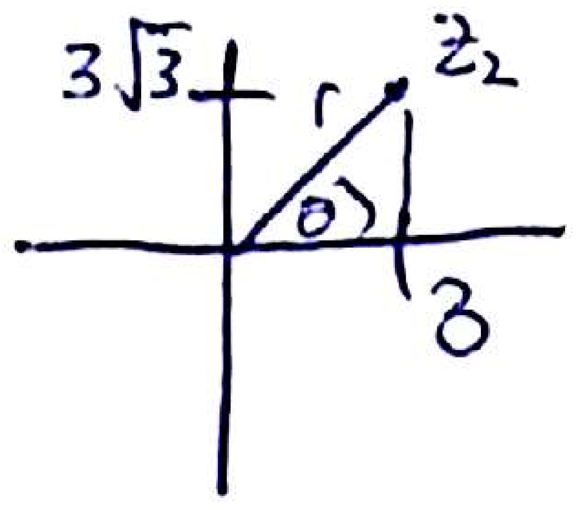
Let $z_1 = a+bi$, $z_2 = c+di$. Then

$$\overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i}$$
$$=(a+c)\cancel{+}-(b+d)i$$
$$=(a-bi)+(c-di)$$
$$=\overline{z_1}+\overline{z_2}.$$

[3] **4.** (a) Express $z = \frac{-22}{3} e^{\frac{\pi}{3} i}$ in polar form.

$$= \frac{22}{3} e^{\frac{4\pi}{3} i} = \frac{22}{3}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

[4] (b) Consider the complex number $z_2 = 3 + i3\sqrt{3}$. Write z_2 in polar and exponential forms.



$$r^2 = 3^2 + (3\sqrt{3})^2$$
$$= 4 \cdot 3^2$$
$$r = \sqrt{4 \cdot 3^2} = 6.$$

$$\sin\theta = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$$
$$\Rightarrow \theta = \frac{\pi}{3}.$$

$$\therefore z_2 = 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 6e^{\frac{\pi}{3} i}$$

[7] (c) Find all cube roots of z_2. Express your answers in exponential form.

$$z_2 = 6e^{\left(\frac{\pi}{3} + 2k\pi\right) i}$$

$$(z_2)^{1/3} = \left(6e^{\left(\frac{\pi}{3} + 2k\pi\right) i}\right)^{1/3} = 6^{1/3} e^{\left(\frac{\pi}{9} + \frac{2}{3}k\pi\right) i}$$

$$k=0: 6^{1/3} e^{\frac{\pi}{9} i}$$
$$k=1: 6^{1/3} e^{\frac{7\pi}{9} i}$$
$$k=2: 6^{1/3} e^{\frac{13\pi}{9} i}$$

[5] **5.** (a) Give bounds of solutions of

$$(2-i)x^{210} - \sqrt{3}x^4 + (1+i\sqrt{2})x^2 + i = 0$$

Let $M = \max\{|-\sqrt{3}|, |1+i\sqrt{2}|, |i|\}$

$= \max\{\sqrt{3}, \sqrt{1^2+\sqrt{2}^2}, 1\}$

$= \max\{\sqrt{3}, \sqrt{3}, 1\} = \sqrt{3}.$

Then if x is any zero, we have

$$|x| < \frac{\sqrt{3}}{|2-i|} + 1 = \frac{\sqrt{3}}{\sqrt{5}} + 1 = \sqrt{\frac{3}{5}} + 1.$$

[5] (b) Let $P(x) = x^{17} - kx + 2$, where k is a real number. When $P(x)$ is divided by $(x-1)$, the remainder is $\frac{-7}{4}$. Find the value of k.

$P(1) = 1^{17} - k(1) + 2 = \frac{-7}{4}.$

$\Rightarrow 1 - k + 2 = \frac{-7}{4}$

$k = 3 + \frac{7}{4} = \frac{19}{4}.$

6. Consider the polynomial

$$P(x) = x^5 - 3x^4 + 3x^3 - 3x^2 + 2x. = x(x^4 - 3x^3 + 3x^2 - 3x + 2$$

[1] (a) How many roots does $P(x) = 0$ have?

5 (counting multiplicities)

[4] (b) Use the rational root theorem to list the possible rational roots of $P(x) = 0$.

If $\frac{p}{q}$ is a rational root of $P(x)$, then

$p|2,\ q|1 \Rightarrow \frac{p}{q} \in \{\pm 1, \pm 2\}$

[4] (c) What do Descartes' Rules of Signs say about $P(x)$? Be specific, justify your answers, and use complete sentences.

$P(x) = x^5 - 3x^4 + 3x^3 - 3x^2 + 2x$

$P(-x) = -x^5 - 3x^4 - 3x^3 - 3x^2 - 2x$

$P(x)$ has 4 sign changes $\Rightarrow$ there are 4, 2, or 0 positive real zeros of $P(x)$

$P(-x)$ has 0 sign changes $\Rightarrow$ there are 0 negative real zeros of $P(x)$.

[7] (d) Using (a), (b) and (c) above, find all the roots of $P(x) = 0$.

$P(x) = x(x^4 - 3x^3 + 3x^2 - 3x + 2)$

$P(1) = 1(1^4 - 3 + \cancel{3} - \cancel{3} + 2) = 0$

$\therefore P(x) = x(x-1)(x^3 - 2x^2 + x - 2)$

$P(2) = 0$

$\therefore P(x) = x(x-1)(x-2)(x^2+1) = x(x-1)(x-2)(x-i)(x+i)$

$\therefore$ The roots of $P(x) = 0$ are

$0, 1, 2, i, -i$.

[2] (e) Write $P(x)$ as a product of linear and irreducible real factors.

$P(x) = x(x-1)(x-2)(x^2+1)$.

[12] **7.** Consider the following matrices

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} -1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}.$$

(a) Indicate if the expression is defined or undefined by placing a check mark in the appropriate column. If it is defined, then indicate the dimensions of the result.

Expression	Defined	Undefined	Dimensions
$A^T(8A) + B$ $2\times3 \quad 3\times2$	✓		2×2
$(AC)^T + CD$ $3\times3 \quad 2\times3$		✓	
$C^TA + 5A$ $3\times2 \quad 3\times2$		✓	

(b) Calculate, if possible, $AB + 2C^T$.

$$\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 3 \\ 5 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 12 & 4 \\ 0 & 4 \\ 6 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 6 \\ 10 & 2 \\ 4 & 10 \end{bmatrix} = \begin{bmatrix} 14 & 10 \\ 10 & 6 \\ 10 & 14 \end{bmatrix}$$

(c) Calculate, if possible, $(1 - i)B$.

$$\begin{bmatrix} 3-3i & 1-i \\ 0 & 2-2i \end{bmatrix}.$$

$$A_{3\times2}X_{2\times4} = \underset{\sim}{0}_{3\times4}$$

(d) Find X such that $AX = \mathbf{0}$ where $\mathbf{0}$ is the 3×4 zero matrix.

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X_{2\times2}B_{2\times2} = B_{2\times2}$$

(e) Find X such that $XB = B$.

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

END OF EXAM