MATH 1210 (Winter Term 2018)

Solutions to the Midterm

Q1. Use the principle of mathematical induction to prove the identity

$$2 + 7 + 12 + 17 + \cdots + (5\,n - 3) = \frac{n\,(5\,n-1)}{2},$$

for $n \geqslant 1$.

Solution: First, assume $n = 1$. Then the LHS (left-hand-side) is 2, and the RHS is $\frac{1.(5\times 1-1)}{2} = 2$. Hence the identity is true for $n = 1$.

Assume it to be true for $n = k$, i.e.,

$$2 + 7 + 12 + 17 + \cdots + (5\,k - 3) = \frac{k\,(5\,k-1)}{2}.$$

Now add $5(k+1) - 3 = 5k + 2$ to both sides. Then

$$\begin{aligned} &2 + 7 + 12 + 17 + \cdots + (5\,k - 3) + (5k+2) \\ &= \frac{k\,(5\,k-1)}{2} + (5k+2) = \frac{5k^2 - k + 10k + 4}{2} = \frac{5k^2 + 9k + 4}{2} \\ &= \frac{(k+1)(5k+4)}{2} = \frac{(k+1)(5(k+1)-1)}{2}, \end{aligned}$$

which proves the identity for $n = k+1$. Hence the identity is proved for all $n \geqslant 1$ by the principle of mathematical induction.

Q2. Express the complex number $(\overline{2\,e^{i\,\pi/3}})^4$ in Cartesian form.

Solution: Let $z = 2\,e^{i\,\pi/3}$. Then $\overline{z} = 2\,e^{-i\,\pi/3}$, and

$$(\overline{z})^4 = 2^4\,e^{-4i\,\pi/3} = 16\,[\cos(-4\pi/3) + i\sin(-4\pi/3)] = 16\,(-1/2 + i\,\sqrt{3}/2) = -8 + i\,8\sqrt{3},$$

which is the required Cartesian form.

Q3. Find all solutions to the equation

$$z^3 = 1 + i.$$

Express your solutions in exponential form.

Solution: Recall that $w = 1 + i$ has three distinct cube-roots. It would be useful to express w in exponential form. It is immediate that $\arg(w) = \pi/4$, and $|w| = \sqrt{2}$, so we may write

$$w = \sqrt{2}\, e^{i(\pi/4+2k\pi)},$$

where k is an arbitrary integer. Since $z^3 = w$, we have

$$z = w^{1/3} = \sqrt[6]{2}\, e^{i\frac{(\pi/4+2k\pi)}{3}} = \sqrt[6]{2}\, e^{i(\pi/12+2k\pi/3)}.$$

Taking $k = 0, 1, 2$, we get the three solutions:

$$z = \sqrt[6]{2}\, e^{i\pi/12}, \quad \sqrt[6]{2}\, e^{9i\pi/12}, \quad \sqrt[6]{2}\, e^{17i\pi/12}.$$

Q4. Consider the polynomial

$$f(x) = x^3 + 4x^2 + kx + 3.$$

It is given that if you divide $f(x)$ by $x + 3$, then the remainder is $k + 1$. Find the value of k.

Solution: By the remainder theorem, we have $f(-3) = k + 1$. After expansion, we get

$$(-3)^3 + 4(-3)^2 + (-3)k + 3 = -27 + 36 - 3k + 3 = 12 - 3k = k + 1,$$

i.e., $11 = 4k$, and hence

$$k = \frac{11}{4}.$$

Q5. Consider the polynomial [6]

$$g(x) = x^3 - t\,x^2 - 1,$$

where t is an integer. Find all values of t for which $g(x)$ has a rational root.

Solution: Let p/q be a possible rational root. By the Rational Roots Theorem, p must divide -1, i.e., $p = \pm 1$. Moreover, q must divide 1, i.e., $q = \pm 1$. Hence the rational root must be ± 1.
If $g(1) = 0$, then $1 - t - 1 = 0 \implies t = 0$. If $g(-1) = 0$, then $-1 - t - 1 = 0 \implies t = -2$. This implies that

$$t = 0, -2,$$

are the values.

Q6. Let

$$A = \begin{bmatrix} -1 & 2 \\ 4 & 5 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ 0 & -7 \\ 5 & 2 \end{bmatrix}.$$

Find the matrix $A^T B - I_2$.

Solution: We have

$$A^T B = \begin{bmatrix} -1 & 4 & 1 \\ 2 & 5 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 0 & -7 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -25 \\ 6 & -37 \end{bmatrix}$$

Hence

$$A^T B - I_2 = \begin{bmatrix} 2 & -25 \\ 6 & -37 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -25 \\ 6 & -38 \end{bmatrix}.$$

Q7. Let P be the plane in $\mathbb{R}^3$ defined by the equation $x+2y-z=3$. Find the parametric equations of the line perpendicular to P which passes through the point $(3,0,3)$.

Solution: The vector $n = \langle 1, 2, -1 \rangle$ is normal to the plane, and we are told that it is a direction vector for the line. Hence the symmetric equations of the line are

$$\frac{x-3}{1} = \frac{y-0}{2} = \frac{z-3}{-1} = t,$$

for some parameter t. Hence the parametric equations are

$$x = t+3, \quad y = 2t, \quad z = -t+3.$$

Q8. Consider the vectors $\mathbf{u} = \langle 1, -1, 2 \rangle$, and $\mathbf{v} = \langle 3, 1, -7 \rangle$ in $\mathbb{R}^3$. Find a vector $\mathbf{w}$, other than the zero vector, such that

$$\mathbf{u} \cdot \mathbf{w} = 0, \quad \text{and} \quad \mathbf{v} \cdot \mathbf{w} = 0.$$

Solution: By construction, the vector $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, and thus has the required property. Hence

$$\mathbf{w} = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 3 & 1 & -7 \end{vmatrix} = \langle 5, 13, 4 \rangle$$

is a solution. Any nonzero multiple of this vector is also a valid solution.

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