The University of Manitoba

MATH 1210: Techniques of Classical and Linear Algebra (Winter Term 2018)

Makeup Midterm Solutions

Q1. Use the principle of mathematical induction to prove the identity [7]

$$3 + 7 + 11 + \cdots + (4n - 1) = n(2n + 1) \qquad \text{for } n \geqslant 1.$$

Proof. When $n = 1$ we have that $3 = 1(2(1)+1)$, so the identity is true when $n = 1$. This will be our base case.
Now assume $3 + 7 + \cdots + (4k - 1) = k(2k + 1)$ for some integer $k \geq 1$. Then

$$\begin{aligned}
3 + 7 + \cdots + (4k - 1) + (4(k + 1) - 1) &= k(2k + 1) + 4k + 3 \\
&= 2k^2 + 2k + 3k + 3 \\
&= (k + 1)(2k + 3) \\
&= (k + 1)(2(k + 1) + 1).
\end{aligned}$$

Therefore if the identity is true for k, it is true for $k+1$. By the principle of mathematical induction, the identity is true for all integers $n \geq 1$. □

Q2. Find all the solutions to the equation [6]

$$z^3 = -8i.$$

Express your solutions in exponential form.

Solution: Converting $-8i$ to polar form gives the equation $z^3 = 8e^{i\frac{3\pi}{2}}$. Finding the three third roots of $8e^{i\frac{3\pi}{2}}$ gives us

$$z = 2e^{i\frac{\pi}{2}}, 2e^{i\frac{7\pi}{6}}, 2e^{i\frac{-\pi}{6}}$$

as all the solutions.

Q3. Let [6]

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} i & -i \\ 1 & i \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Find the matrix $AB + C^T$.
Solution: We have

$$AB + C^T = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & i \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3i+2 & -i \\ 1-i & 2i \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3i+3 & -2-i \\ 3-i & 2i+1 \end{bmatrix}.$$

So $AB + C^T = \begin{bmatrix} 3i+3 & -2-i \\ 3-i & 2i+1 \end{bmatrix}$.

Q4. Consider the polynomial [6]

$$f(x) = x^4 + 3x^3 + kx^2 + 2kx + 3.$$

It is given that if you divide $f(x)$ by $x+1$, then the remainder is k. Find the value of k.
Solution: By the remainder theorem we have

$$k = f(-1) = 1 - 3 + k - 2k + 3 = 1 - k$$

Solving for k in the equation $k = 1 - k$ gives $k = \frac{1}{2}$.

Q5. Express the complex number $\overline{(3e^{i\pi/6})^3}$ in Cartesian form. [5]
Solution: We have

$$\overline{(3e^{i\pi/6})^3} = \overline{27e^{i\pi/2}} = 27e^{-i\pi/2}$$

which in cartesian form is $-27i$.

Q6. Find the equation of a plane passing through the point $(1, 2, 1)$ and the line given by the parametric equations [9]

$$x = t$$
$$y = -1 + t$$
$$z = -2 + t$$

for t a real number.
Solution: We already have a point on the plane, so we need a normal vector. To find it, we will find two vectors parallel to the plane and take their cross product. The direction vector of the line $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is one such vector. Letting $t = 0$, we find a point

$Q = (0, -1, -2)$ on the line and therefore on the plane. The vector $\mathbf{w}$ joining the point given $P = (1, 2, 1)$ and $Q = (0, -1, -2)$ is $\mathbf{w} = \left(\begin{smallmatrix}1\\3\\3\end{smallmatrix}\right)$.
Now $\mathbf{v}$ and $\mathbf{w}$ are two vectors parallel to the plane, and their cross product is given by $\mathbf{v} \times \mathbf{w} = \left(\begin{smallmatrix}0\\-2\\2\end{smallmatrix}\right)$. Therefore we may let $\mathbf{n} = \left(\begin{smallmatrix}0\\-1\\1\end{smallmatrix}\right)$ be the normal vector to the plane.
Given a normal vector $\mathbf{n}$ and a point $P = (1, 2, 1)$ we can find the equation of the plane by

$$\mathbf{n} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right) = \mathbf{0}.$$

Expanding this out gives the equation of the plane as $-y + z = -1$.

Q7. Consider the polynomial [6]

$$g(x) = x^3 + tx^2 + 4x + 1,$$

where t is an integer. Find all values of t for which $g(x)$ has a rational root.
Solution: The rational roots theorem says that all the possible roots of $g(x)$ are 1 and -1 . For $x = 1$ to be a root we must have $0 = g(1) = 1 + t + 4 + 1$ so $t = -6$. For $x = -1$ to be a root we must have $0 = g(-1) = -1 + t - 4 + 1$ so $t = 4$. Therefore all the values of t for which $g(x)$ has a rational root are $t = -6, 4$.

Q8. Find the angle θ between the vectors $\mathbf{v} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$. Give you answer with $0 \leq \theta \leq \pi$. [5]
Solution: We know

$$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}||\mathbf{u}|} = \frac{-2}{\sqrt{8}} = -\frac{1}{\sqrt{2}}.$$

The only value of θ such that $\cos(\theta) = -\frac{1}{\sqrt{2}}$ and $0 \leq \theta \leq \pi$ is $\theta = 3\pi/4$.