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1. Let $\mathbf{u}, \mathbf{w} \in \mathbb{R}^3$ be the vectors

$$u = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \qquad w = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}.$$

Let $\mathbf{v} \in \mathbb{R}^3$ be the vector

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We would like to find the numbers x, y, and z, which solve the equation

$$(\mathbf{u} \times \mathbf{v}) + \mathbf{v} = \mathbf{w}. \tag{1}$$

[5] (a) Show that Equation (1) can be written as the linear system

$$\begin{aligned} x + 2z &= -1, \\ y - 2z &= 5, \\ -2x + 2y + z &= 3. \end{aligned}$$

[10] (b) Solve the system of equations for $\mathbf{v}$.

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[6] 2. For this problem, you may only use the rules for working with complex numbers (and exponentials) and the formula

$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix}).$$

Let $t \in \mathbb{R}$ be an arbitrary real number. Simplify

$$2[\cos(\pi t)]^2 - 1.$$

Do *not* use any trig identities.

[5] 3. Solve the following matrix equation for the matrix X:

$$-5X - 2\begin{bmatrix} -5 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

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[11] 4. Fill in the blanks to complete the proof using the principle of induction that for all positive integers n, if A is an upper triangular matrix of size $n \times n$, then the determinant of A is the product of its diagonal elements.

We use the following notation: If A is a matrix of size $(k+1) \times (k+1)$, then

- a_{ij} is the entry in the i-th row and the j-th column of A; and
- A_{ij} is the matrix obtained by deleting the i-th row and the j-th column of A; and
- c_{ij} is the cofactor of the (i,j)-th entry of A.

The product of the diagonal elements of A can then be written as $a_{11}a_{22}\ldots a_{nn}$.

Proof: For all integers $n \geqslant 1$, let P_n be the statement:

(a)

Base Case: The statement P_1 says that if A is an upper triangular 1×1 matrix, then the determinant of A is the product of its diagonal elements. This is true.

Induction step: Let $k \geqslant 1$ be an arbitrary but fixed integer. Assume that the statement P_k holds, i.e., if A is an upper triangular matrix of size $k \times k$, then

(b)

Now, we wish to prove the statement P_{k+1}, i.e. if A is an upper triangular matrix of size $(k+1) \times (k+1)$, then

(c)

To that end, we suppose that A is an arbitrary upper triangular matrix of size $(k+1) \times (k+1)$. We expand with respect to the first column, and use the fact that the only non-zero entry in the first column is a_{11}:

(d)

$$|A| = \sum_{i=1}^{n} a_{i1} c_{i1} = \underline{\hspace{6cm}} \tag{2}$$

The cofactor c_{11} is by definition ($(-1)^2$ times) the determinant of the matrix A_{11}. Thanks to P_k we know that

(e)

$$c_{11} = |A_{11}| = \underline{\hspace{6cm}} \tag{3}$$

(f) Thus, combining Equation _ with Equation _ we obtain

$$|A| = \underline{\hspace{1cm}}$$

(g) Thus P_{k+1} holds. By ___ we conclude that the statement P_n holds for all positive integers n.

[0] 5. **BONUS: MAX 3 MARKS**. Let A be a square matrix and $n \in \mathbb{Z}^+$ be such that $A^n = I$. Show that A must be invertible and find the inverse.