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[6] 1. Use the formulas (valid for all $t \in \mathbb{R}$),

$$\cos(t) = \frac{1}{2}\left(e^{it} + e^{-it}\right), \qquad \sin(t) = \frac{i}{2}\left(e^{-it} - e^{it}\right), \tag{1}$$

to calculate

$$2\sin(\pi t)\cos(\pi t).$$

Any solution in which the formulas in equation (1) are not used will be given a zero. Your answer must be of the form $\sin(\lambda t)$ for some $\lambda \in \mathbb{R}$. (You don't need to show that the formulas in Eq. (1) are valid.)

Solution: We start by substituting Eq. 1:

$$\begin{aligned} 2\sin(\pi t)\cos(\pi t) &= 2\left(\frac{i}{2}\left(e^{-i\pi t} - e^{i\pi t}\right)\right)\left(\frac{1}{2}\left(e^{i\pi t} + e^{-i\pi t}\right)\right) \\ &= \frac{i}{2}\left(e^{-i\pi t} - e^{i\pi t}\right)\left(e^{i\pi t} + e^{-i\pi t}\right) \end{aligned}$$

We then expand the brackets:

$$\begin{aligned} \frac{i}{2}\left(e^{-i\pi t} - e^{i\pi t}\right)\left(e^{i\pi t} + e^{-i\pi t}\right) &= \frac{i}{2}\left(1 - 1 + e^{-2i\pi t} - e^{2i\pi t}\right) \\ &= \frac{i}{2}(e^{-2i\pi t} - e^{2i\pi t}) \\ &= \sin(2\pi t). \end{aligned}$$

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[9] 2. Let A be an upper triangular matrix. Use the principle of induction to prove that for all positive integers n, A^n is upper triangular. You may use (without proving it) the fact that if B and C are upper triangular matrices of the same size, then BC is also upper triangular.

Solution: For all positive integers n, let P_n be the statement: "A^n is upper triangular".

Base case: The statement P_1 says that A^1 is upper triangular. Because $A^1 = A$ this is true.

Inductive step: Let k be an arbitrary but fixed positive integer. Assume that the statement P_k holds, i.e., the matrix A^k is upper triangular. We must then prove that P_{k+1} holds, i.e. A^{k+1} is upper triangular. To that end, we write $A^{k+1} = A^k A$, thus A^{k+1} is the product of two upper triangular matrices, which implies that it is upper triangular itself. We conclude that P_{k+1} holds.

Conclusion: We have shown that P_1 is true, and that P_k implies P_{k+1} for all positive integers k, thus, by the principle of mathematical induction, the statement P_n holds for all positive integers n.

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[6] 3. Given that $z_1 = 3 + i$ is a zero of $f(x) = x^4 - 6x^3 + 13x^2 - 18x + 30$, find all zeros of $f(x)$.

Solution: We use the strategy used for problems 10 & 11 from Section 3.1 in the textbook.

Because f has only real coefficients, we know that $\bar{z}_1 = 3 - i$ is a zero of f as well. This means that the polynomial $(x - z_1)(x - \bar{z}_1)$ must be a factor of f. We expand it:

$$\begin{aligned} g(x) &= (x - z_1)(x - \bar{z}_1) = (x - (3 + i))(x - (3 - i)) \\ &= x^2 - \big((3 - i) + (3 + i)\big)x + (3 + i)(3 - i) \\ &= x^2 - 6x + 10 \end{aligned}$$

Now, we can use e.g. long division to find the polynomial $h(x)$ such that $f(x) = g(x)h(x)$:

$$\begin{array}{r|rrrrr} & & & x^2 & & +3 \\ \hline x^2 - 6x + 10 & x^4 & -6x^3 & +13x^2 & -18x & +30 \\ & -x^4 & +6x^3 & -10x^2 & & \\ \cline{2-4} & & & 3x^2 & -18x & +30 \\ & & & -3x^2 & +18x & -30 \\ \cline{4-6} & & & & & 0 \end{array}$$

The result is $h(x) = x^2 + 3$. The zeroes of h are $\pm\sqrt{3}i$. Thus, the zeroes of the original polynomial f are:

$$3 + i, \qquad 3 - i, \qquad \sqrt{3}i, \qquad -\sqrt{3}i.$$

4. Consider the matrix

$$B = \begin{pmatrix} 0 & -a & -b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$$

[6] (a) Calculate B^n for all integers $n \geqslant 2$. Your answer should be presented in the form:

$$B^n = \begin{cases} \ldots & \text{if } n = 2 \\ \ldots & \text{if } n = 3 \\ \ldots & \text{if } n \geqslant 4 \end{cases}$$

Solution: First we calculate B^2:

$$B^2 = \begin{pmatrix} 0 & -a & -b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -a & -b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Next, we calculate B^3

$$B^3 = B^2 B = \begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -a & -b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now, if $n \geqslant 4$, then $B^n = B^3 B^{n-3}$, and because B^3 is the zero matrix, it follows that B^n is the zero matrix. (We need $n \geqslant 4$ for B^{n-3} to make sense!)

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[0] (b) **BONUS: MAX 4 MARKS.** Calculate the matrix D given by the formula

$$D = \mathrm{I}_3 + \sum_{n=1}^{5} B^n.$$

Let A be the matrix $A = \mathrm{I}_3 - B$, then calculate AD.

Solution: The matrix D is given by

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -a & -b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$$

$$AD = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -a+a & -b+ac-ac+b \\ 0 & 1 & -c+c \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{I}_3$$