
Attempt all questions and show all your work. Some or all questions will be marked. Simplify your answers as much as possible.

1. Consider point $P(2,1,-3)$, plane $\Pi\ :\ 4x-y+3z=-2$ and line

$$\ell\ :\ x=1+2t\,,\quad y=-1-2t\,,\quad z=2+4t,\quad \text{where}\quad t\in\mathbf{R}.$$

   (a) Is the point P on the plane Π? Why?

   (b) Find the intersection point of the plane Π and the line ℓ.

   (c) Find parametric equations and symmetric equations of the line through P and perpendicular to the plane Π.

   (d) Find an equation for the plane containing the point P and the line ℓ.

   (e) Let ℓ_1 be the line passing through the point P and the origin. Given that the lines ℓ and ℓ_1 intersect, find the intersection point.

2. Solve the first two systems of linear equations using Gauss-Jordan elimination method and the third one by Gaussian elimination method.

$$(i)\quad \begin{aligned} 2x+6y+2z &= -1\\ x+3y+w &= 4\\ +z+w &= -1 \end{aligned} \qquad\qquad (ii)\quad \begin{aligned} 4x+6y-z &= 8\\ 2x+3y &= 5\\ 2x+2y-3z &= 4\\ 4x+4y-11z &= 1 \end{aligned}$$

$$(iii)\quad \begin{aligned} 2x_1-4x_2-3x_3+12x_4 &= 1\\ -x_1+2x_2+2x_3-7x_4 &= 1\\ -x_1+2x_2-3x_4 &= -5 \end{aligned}$$

3. Consider the homogeneous system of linear equations

$$\begin{aligned} 2x_1+10x_2-8x_3+2x_6 &= 0\\ 3x_1+15x_2-12x_3+x_5+2x_6 &= 0\\ -x_1-5x_2+4x_3+x_4+x_6 &= 0 \end{aligned}$$

   (a) Without solving predict the number of solutions of this system. Explain.

   (b) Find all **basic** solution.

4. Use Cramer's rule to solve the following system of linear equations for x_3 ***only***.

$$\begin{aligned}
-x_1 + 6x_2 \qquad\quad + 3x_4 &= -5\\
x_2 \qquad\quad + x_4 &= -1\\
4x_1 - x_2 \qquad\quad + 2x_4 &= -5\\
5x_2 \quad + x_3 \qquad\quad &= 0
\end{aligned}$$

5. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 5 & 1 \\ 6 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 1 & 5 & 0 \end{bmatrix}$. Find determinant of the matrix $A^{2022}B^{2022} - 2A^{2021}B^{2021}$.

6. Determine whether each set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly dependent or linearly independent.
   (a) $\mathbf{u}_1 =< -10, 11 >, \mathbf{u}_2 =< 3, 15 >, \mathbf{u}_3 =< -14, 21 >$
   (b) $\mathbf{u}_1 =< 6, 3, 4 >, \mathbf{u}_2 =< 1, 7, 5 >, \mathbf{u}_3 =< 5, -4, -1 >$
   (c) $\mathbf{u}_1 =< 1, 2, 4, -2 >, \mathbf{u}_2 =< 1, 3, 4, -3 >, \mathbf{u}_3 =< 1, 2, 4, -1 >$

7. Let $\mathbf{u} =< 1, 2, -3, 0, 1 >$, $\mathbf{v} =< 4, -1, 0, 0, -2 >$ and $\mathbf{w} =< -6, 3, -2, a - 2b, 2a >$ where a and b are integers. Find values of a and b such that $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.