MATH 1210 Final Exam Winter 2023

Date and Time: April 14 at 6:00 PM   Duration: 2 Hours

Attempt all questions and show your work. Simplify your answers as much as possible.

1. (7 points) Let $A = \begin{bmatrix} 1 & -i \\ 0 & 1+i \end{bmatrix}$ where i the complex number for which $i^2 = -1$. Use mathematical induction on the integer $n \geq 1$ to prove that

$$A^n = \begin{bmatrix} 1 & 1-(1+i)^n \\ 0 & (1+i)^n \end{bmatrix}.$$

2. Consider the polynomial $P(x) = x^3 + x^2 + 2x + 2$.
   (a) (3 points) Prove that $P(x)$ has no zeros in the interval $[0\ \ 2]$.
   (b) (6 points) Find all the zeros of $P(x)$.

3. Consider the two lines

$$\begin{aligned} \ell_1 : &\quad x = 1+t\,, \quad y = -2+t\,, \quad z = 3-2t \\ \ell_2 : &\quad x = 2+s\,, \quad y = 3-3s\,, \quad z = -2+s\,. \end{aligned}$$

   (a) (5 points) Find the intersection point of the lines ℓ_1 and ℓ_2.
   (b) (5 points) Find an equation of the plane that contains the two lines ℓ_1 and ℓ_2.

4. Consider the system

$$\begin{aligned} x_1 + x_2 - 2x_3 + 4x_4 &= 0 \\ -2x_1 - x_2 + 4x_3 - 5x_4 &= 0 \\ 4x_1 + x_2 - 8x_3 + 7x_4 &= 0. \end{aligned}$$

   (a) (3 points) Without solving the system, show that it cannot have a unique solution.
   (b) (9 points) Use Gauss-Jordan elimination to find all basic solutions of the system.

5. (8 points) Consider the system

$$\begin{aligned} 2x + z &= 3 \\ x + 2y + tz &= 5 \\ -x + 2y &= 4 \end{aligned}$$

   where t is a real number. It is given to you that the system has a unique solution (x, y, z) with $z = 1$. Use Cramer's rule to find t.

6. Let $\mathbf{u}_1 = \langle 1, 0, 0, 0 \rangle$, $\mathbf{u}_2 = \langle 2, 1, -3, -2 \rangle$ and $\mathbf{u}_3 = \langle 4, 2, -5, -1 \rangle$.
   (a) (6 points) Determine if the vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ are linearly dependent or linearly independent.
   (b) (6 points) Write the vector $\mathbf{v} = \langle 0, -1, 2, -1 \rangle$ as a linear combination of the vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

7. Consider the invertible matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 2 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

   It is given that $A^{-1} = \begin{pmatrix} b_{11} & b_{12} & \frac{3}{2} \\ b_{21} & b_{22} & 2 \\ b_{31} & b_{32} & 1 \end{pmatrix}$.

(a) (5 points) Find the entry b_{21} of A^{-1} using the adjoint formula.

(b) (3 points) Solve the linear system $AX + 2B = \mathbf{0}$, where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{0}$ is the zero matrix of the appropriate size.

8. Let $\mathbf{u} = \langle 1, 0 \rangle$, $\mathbf{v} = \langle -2, 1 \rangle$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(\mathbf{u}) = \langle -1, 1 \rangle$ and $T(\mathbf{v}) = \langle 1, 2 \rangle$.
   (a) (4 points) Use linearity of T to find $T(\langle 0, 1 \rangle)$.
   (b) (3 points) Find the matrix associated with T.
   (c) (2 points) Find the image of $\langle 5, -2 \rangle$ under T.

9. (7 points) Find all eigenvalues of the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ -1 & 0 & -1 \end{pmatrix}$.

10. (8 points) Consider the matrix $A = \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix}$. Given that one of the eigenvalues of A is $1 + i$, find all eigenvectors that correspond to the eigenvalue $1 + i$.