MATH 1210 Midterm Exam Winter 2023

Date and Time: March 14 at 5:45 PM Duration: 75 minutes

Attempt all questions and show your work. Simplify your answers as much as possible.

1. (9 points) Use mathematical induction to prove that

$$(1-\frac{1}{4})(1-\frac{1}{6})(1-\frac{1}{8})\cdots(1-\frac{1}{2n}) = \frac{(2n)!}{2^{2n-1}(n!)^2}$$

for all integers $n \geq 2$. (You are **not** allowed to use any other method.)

2. (6 points) Use the identities

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

to evaluate the following sum

$$\sum_{k=11}^{20} \left[(k-10)^2 - 2k\right].$$

3. Let $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i$.

   (a) (5 points) Show that z is one the 50^{th} roots of the complex number i.
   Hint: You do not need to find all 50^{th} roots of i.

   (b) (8 points) Evaluate the following expression. Write your answer in Cartesian form and simplify as much as possible.

$$\frac{z^{50}}{\left(z - \frac{2}{\sqrt{2}}\,i\right)^{60}}$$

4. Let $P(x) = 8x^4 - 4x^3 + 10x^2 + x + 5$.

   (a) (4 points) Use the Rational Roots Theorem to find all possible rational roots of $P(x)$.

   (b) (4 points) Apply the Bounds Theorem to $P(x)$ and use it to eliminate some possible roots from the list obtained in (a).

   (c) (6 points) Given that $2x - 1 + 2i$ is a factor of $P(x)$, find all roots of $P(x)$.

5. Consider the matrices

$$A = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 1 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \\ 5 & -1 \end{bmatrix}.$$

   (a) (6 points) Evaluate the expression $CA^2 - 2(AB)^T$.

   (b) (3 points) Suppose that E is a matrix such that the expression $B^T - EC$ is defined. Find the size of E.

6. Consider the point $P(0,1,-1)$ and the vectors $\mathbf{u} = \langle k, 5, 2\rangle$ and $\mathbf{v} = \langle k, -k, 3\rangle$.

   (a) (3 points) For which values of k are the vectors $\mathbf{u}$ and $\mathbf{v}$ orthogonal?

   (b) (6 points) For $k = 1$, find an equation of the plane that contains the point P and its normal vector is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

7. (10 points) Solve the following linear system of equations.

$$\begin{array}{rrrrl} 7x_1 & +x_2 & & +x_4 & = 1 \\ 6x_1 & +x_2 & & & = 0 \\ 5x_1 & +x_2 & & -x_4 & = -1 \\ & & x_3 & +2x_4 & = 1 \end{array}$$