MATH 1210 Assignment 3 Winter 2024 **Due date: April 1, 11:59 PM**

Attempt all questions (note that selected questions will be marked), show all your work and justify your answers (unless it is explicitly stated that you do not have to do that). Unjustified answers will receive LITTLE or NO CREDIT. Also, no partial credit will be given if you do not use the method specified in the question.

1. Consider the lines

$$\ell_1: \quad x = 1 + 2t, \quad y = 1 - t, \quad z = 3t, \quad t \in R,$$

$$\ell_2: \quad \frac{x-1}{2} = \frac{y+2}{-1} = z - 5,$$

the plane

$$\Pi: \quad x + 2y + 3z + 3 = 0,$$

and the point $P(1, 0, 2)$.

(a) Find the point Q of intersection of line ℓ_2 and the plane Π.

(b) Either find an equation of the plane which is perpendicular to the plane Π and passes through the points P and Q, or explain why such a plane does not exist.

(c) Do the lines ℓ_1 and ℓ_2 intersect each other? If so, find the point of their intersection.

(d) Are the lines ℓ_1 and ℓ_2 parallel to each other? Explain.

(e) Either find an equation of the plane containing the line ℓ_1 that is also parallel to the line ℓ_2, or explain why such a plane does not exist.

(f) Either find an equation of the plane containing the line ℓ_1 that is also perpendicular to the line ℓ_2, or explain why such a plane does not exist.

(g) Find both parametric and symmetric equations (if possible) for **all** lines which are parallel to the plane Π and pass through the point P.

2. Consider the following linear system of equations.

$$\begin{cases} x - 3y + z - 2w = 1 \\ -x + 3y + z + w = -1 \\ -x + 3y + 3z - w = 2 \\ x - 3y - 13z + 8w = -8 \end{cases}$$

(a) Find the reduced row-echelon form (RREF) of the augmented matrix.

(b) Find all solutions of this system (i.e., determine the solution set).

3. Consider the system

$$\begin{cases} x + y + bz &= 2 \\ x + by + 4z &= 1 \\ ax + ay + 2z &= a \end{cases}$$

In each case, determine all values of a and b which give the indicated number of solutions, if possible. If no such a and b exist, give the reason why not.

(i) no solutions

(ii) exactly one solution

(iii) exactly two solutions

(iv) infinitely many solutions

4. Find all basic solutions to the following homogeneous system and express the general solution as a linear combination of these basic solutions:

$$\begin{cases} -2x_1 + 2x_2 - x_3 + 4x_4 + x_5 = 0 \\ 2x_1 - 2x_2 + 2x_3 - 2x_4 = 0 \\ 2x_1 - 2x_2 + 3x_3 + x_5 = 0 \end{cases}$$

5. Use Cramer's rule to solve for y without solving for any of the other variables:

$$\begin{cases} 3x + y - z = -1 \\ 2x - y - 2z = 6 \\ 3x + 2z = -7 \end{cases}$$

6. Suppose that A and B are 2024×2024 matrices such that $AB^2 = -BA^2$ and $\det(A) \neq \det(B)$. Show that either $\det(A) = 0$ or $\det(B) = 0$.

7. Show that $\det(AB^T + A^T) = \det(BA^T + A)$ for any $n \times n$ matrices A and B.

8. Determine whether each of the following sets of vectors is linearly dependent or linearly independent. If the set is linearly dependent, express one of the vectors in this set as a linear combination of the other vectors in this set.

   (a) $\langle 1, -1\rangle$, $\langle 11, 12\rangle$, $\langle 4, 7\rangle$

   (b) $\langle 1, -1, 1\rangle$, $\langle 2, 1, 2\rangle$, $\langle 1, 3, -2\rangle$

   (c) $\langle 1, -1, 1, 0\rangle$, $\langle 5, 1, 5, 2\rangle$, $\langle 2, 1, 2, 1\rangle$

   (d) $\langle 11, 21, 3, 4, 51\rangle$, $\langle 2, 1, 2, 1, -13\rangle$, $\langle 0, 0, 0, 0, 0\rangle$, $\langle -29, 1, 2, 1, -31\rangle$

9. For each of the following matrices find, if possible, the inverse.

   (a) $\begin{pmatrix} 1 & 3 & 6 \\ 2 & 0 & -1 \\ -1 & -1 & -2 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 0 & 1 \end{pmatrix}$

10. Prove or disprove the following statements:

    (a) If A and B are singular matrices of the same size, then $A^2 + B^2$ is also singular.

    (b) If A and B are singular matrices of the same size, then $AB(A^2 + B^2)$ is also singular.

    (c) If A and B are square non-singular matrices of the same, then AB cannot be the zero matrix.