CHAPTER TWO

PLANES AND LINES IN R^3

2.1 INTRODUCTION

In this chapter we will use vector methods to derive equations for planes and lines in three-dimensional space $\mathbf{R}^3$. The derived equations will be vector equations that we will be able convert into non-vector form equations. We will also use vectors to find the distance between a point and a plane, a point and a line, and between two lines in space as well as to find the angle between two planes and between two lines in $\mathbf{R}^3$.

2.2 PLANES IN R^3

A plane is uniquely determined in three-dimensional space by either:
1. giving the orientation of the plane in space and giving a point on the plane; or
2. giving three noncollinear points on the plane.

The orientation of a plane in space is specified by giving a vector **n** that is orthogonal (perpendicular) to the plane. Such a vector is said to be a **normal vector** to the plane. The equation for the plane derived using the normal vector **n** and a point P on the plane is called the point-normal form of the equation of the plane.

The point-normal form equation for a plane

Let $\mathbf{n} = (a, b, c)$ be a normal vector to a plane and let the point $P = (x_0, y_0, z_0)$ be a point on the plane. If $X = (x, y, z)$ is any other point on the plane, then, since $\mathbf{n} \perp \overrightarrow{PX}$ we have $\mathbf{n} \bullet \overrightarrow{PX} = 0$. Let $\mathbf{p} = \overrightarrow{OP}$ and let $\mathbf{x} = \overrightarrow{OX}$. Then $\overrightarrow{PX} = \overrightarrow{PO} + \overrightarrow{OX} = -\overrightarrow{OP} + \overrightarrow{OX} = \overrightarrow{OX} - \overrightarrow{OP} = \mathbf{x} - \mathbf{p}$ so the equation $\mathbf{n} \bullet \overrightarrow{PX} = 0$ becomes $\mathbf{n} \bullet (\mathbf{x} - \mathbf{p}) = 0$. This vector equation for the plane is called the **point -normal form**.
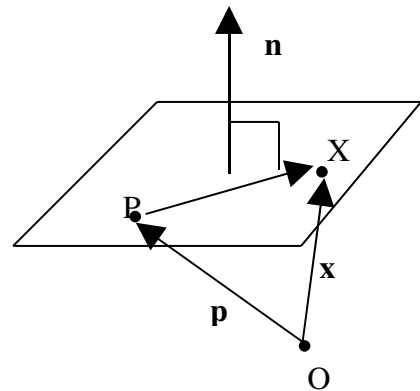
For the nonvector form of this equation we use $\mathbf{n} = (a, b, c)$ and $\mathbf{x} - \mathbf{p} = (x - x_0, y - y_0, z - z_0)$. Then $\mathbf{n} \bullet (\mathbf{x} - \mathbf{p}) = 0$ becomes $(a, b, c) \bullet (x - x_0, y - y_0, z - z_0) = 0$

$$\text{or} \quad a(x-x_0) + b(y-y_0) + c(z-z_0) = 0.$$

If we write $\mathbf{n}\bullet(\mathbf{x}-\mathbf{p}) = 0$ in the form $\mathbf{n}\bullet\mathbf{x} = \mathbf{n}\bullet\mathbf{p}$ and then let $\mathbf{n}\bullet\mathbf{p} = d$ we get $(a,b,c)\bullet(x, y, z) = d$ or in nonvector form $ax + by + cz = d$. This nonvector form equation is called the **standard form** equation for a plane.
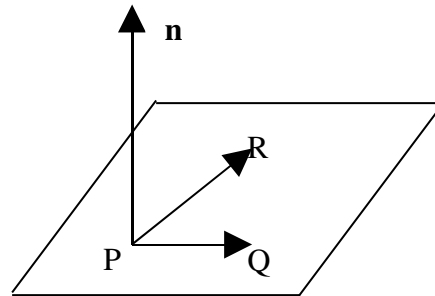
Example

If a plane has the normal vector $\mathbf{n} = (1, 2, 3)$ and passes through the point $P = (4, 5, 6)$, then its point-normal form equation is $(1, 2, 3)\bullet(x-4, y-5, z-6) = 0$.

The standard form equation of this plane is found by computing the dot product above to get $(1)(x-4) + (2)(y-5) + (3)(z-6) = 0$ which when expanded becomes $x-4+2y-10+3z-18 = 0$. This simplifies to the standard form equation of the plane $x+2y+3z = 32$.

Plane determined by three noncollinear points

Let P, Q and R be three noncollinear points lying on a plane. To find an equation for the plane we make use of the cross product. The vector $\mathbf{n} = \overrightarrow{PQ}\times\overrightarrow{PR}$ is a normal vector to the plane. Using this normal vector with any one of the three points on the plane, say P, and using the point-normal form equation from the preceding section, will result in an equation of the plane. The following example illustrates the procedure.

Example A plane passes through three noncollinear points $P = (1, 2, 0)$, $Q = (3, 0, 1)$ and $R = (4, 2, 3)$. Find a standard form equation of the plane.

Solution

$\overrightarrow{PQ} = (3, 0, 1) - (1, 2, 0) = (2, -2, 1)$

$\overrightarrow{PR} = (4, 2, 3) - (1, 2, 0) = (3, 0, 3)$

$\mathbf{n} = \overrightarrow{PQ}\times\overrightarrow{PR} = \left(\begin{vmatrix} -2 & 1 \\ 0 & 3 \end{vmatrix}, -\begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix}, \begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} \right) = (-6, -3, 6)$

$\mathbf{n}\bullet(\mathbf{x}-\mathbf{p}) = 0 \quad \Rightarrow \quad (-6, -3, 6)\bullet(x-1, y-2, z-0) = 0$

$\Rightarrow \quad -6x+6-3y+6+6z = 0$

$$\Rightarrow -6x - 3y + 6z = -12$$
$$\Rightarrow 2x + y - 2z = 4$$

2.2 Problems

1. Find a normal vector to each of the following planes.
   a. $2x + 3y + 4z = 5$ b. $x + 3y - 2z = 6$ c. $3x + 5z = 7$

2. Find a point on the given plane, then write a point-normal form equation for the given plane.
   a. $3x + 2y + 3z = 6$ b. $x + 2y + 3z = 12$ c. $5x + 2y - 3z = 7$

3. Find an equation for the plane passing through the point P and having the vector **n** as a normal vector. Write the equation of the plane in standard form.
   a. P = (5, 3, 4)**n** = (2, 3, 1)   b. P = (4, 1, 2)   **n** = (5, 2, 3)
   c. P = (2, 1, 6)**n** = (1, 0, 1)   d. P = (1, 0, –2)   **n** = (2, –1, 3)

4. Find an equation for the plane passing through the 3 noncollinear points P, Q and R. Write the equation of the plane in standard form.
   a. P = (1, 2, 3) Q = (2, 1, 4)   R = (2, 3, 6)
   b. P = (3, 2, 5) Q = (2, 3, 6)   R = (4, 7, 5)
   c. P = (1, 0, 1) Q = (1, 1, –1) R = (2, 1, –2)
   d. P = (2, 1, 0) Q = (3, 1, 2)   R = (0, –3, 4)

2.3 LINES IN R^3

A line is uniquely determined in three-dimensional space by either:
1. giving the orientation of the line in space and giving a point on the line;   or
2. giving two noncoincident points on the line.

The orientation of a line in space is given by specifying a vector in $\mathbf{R}^3$ that is parallel to the line. The form of the equation derived using a vector parallel to the line and a point on the line is called the point-parallel form equation for a line.
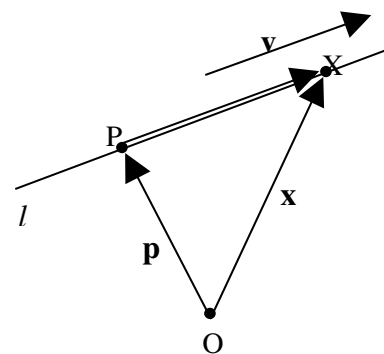
Point-parallel form equation for a line

Let $P = (x_0, y_0, z_0)$ be a point on the line l and let the vector $\mathbf{v} = (a, b, c)$ be a vector that is parallel to the given line l. If $X = (x, y, z,)$ is any other point on the line l, then $\overrightarrow{PX} = t\mathbf{v}$ for some scalar t. But $\overrightarrow{PX}$ can be written as $\overrightarrow{PX} = \overrightarrow{PO} + \overrightarrow{OX} = -\overrightarrow{OP} + \overrightarrow{OX} = \overrightarrow{OX} - \overrightarrow{OP} = \mathbf{x} - \mathbf{p}$, hence $\overrightarrow{PX} = t\mathbf{v}$ becomes $\mathbf{x} - \mathbf{p} = t\mathbf{v}$ or $\mathbf{x} = \mathbf{p} + t\mathbf{v}$. The vector equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$. is called the **point-parallel form equation** for a line in $\mathbf{R}^3$. Each different value of the scalar t corresponds to a different point located on the line.

To obtain the nonvector form parametric equations for a line we start with $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ and write out the vectors as ordered triples. This gives $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$. Equating the corresponding components on each side produces three equations. These equations are called the **parametric equations** of the line and the scalar t is called the **parameter**.
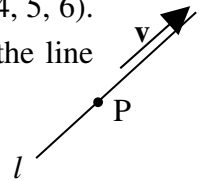
$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

Example

Suppose the line l passes through the point $P = (1, 2, 3)$ and is parallel to the vector $\mathbf{v} = (4, 5, 6)$. Its point-parallel form equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ is $(x, y, z) = (1, 2, 3) + t(4, 5, 6)$. Equating the corresponding components we get as the parametric equations of the line the three equations: $x = 1 + 4t, \quad y = 2 + 5t, \quad z = 3 + 6t$.
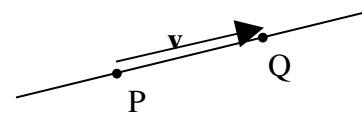
Two point form equation for a line

If the line l passes through two distinct points $P = (x_0, y_0, z_0)$ and $Q = (x_1, y_1, z_1)$ we use the point-parallel form with $\mathbf{v} = \overrightarrow{PQ} = (x_1 - x_0, \ y_1 - y_0, \ z_1 - z_0)$. Equating the corresponding components in the vector equation $(x, y, z) = (x_0, y_0, z_0) + t(x_1 - x_0, y_1 - y_0, z_1 - z_0)$ results in the nonvector **two-point form** parametric equations:

$$x = x_0 + t(x_1 - x_0)$$
$$y = y_0 + t(y_1 - y_0)$$
$$z = z_0 + t(z_1 - z_0)$$

Example

The line l passes through the points P = (1, 2, 3) and Q = (4, 6, 9). The two-point vector form of its equation is

$$(x, y, z) = (1, 2, 3) + t(4 - 1, 6 - 2, 9 - 3)$$

$$\text{or} \quad (x, y, z) = (1, 2, 3) + t(3, 4, 6).$$

Equating components, the corresponding parametric equations are found to be

$$x = 1 + 3t$$
$$y = 2 + 4t$$
$$z = 3 + 6t$$

Symmetric equations for a line

Consider the parametric form equations for a line $x = x_0 + at,\ y = y_0 + bt$ and $z = z_0 + ct$. If $a,\ b$ and c are all nonzero, we can solve each equation for t to get:

$$\frac{x - x_0}{a} = t, \qquad \frac{y - y_0}{b} = t, \qquad \frac{z - z_0}{c} = t.$$

Eliminating t between theses equations gives the **symmetric form** equations for a line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

If any of $a,\ b$ or c equals 0, then the corresponding numerator must equal 0.   Thus for example if $a = 0$, the symmetric equations become $x - x_0 = 0$ and $\dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$.

Example

Find a set of symmetric equations for the line that passes through the two points P = (3, 2, 1) and Q = (4, 5, 6).

Solution

The vector $\mathbf{v} = \overrightarrow{PQ} = (4 - 3, 5 - 2, 6 - 1) = (1, 3, 5)$ is parallel to the required line. Setting $(a, b, c) = (1, 3, 5)$ and $(x_0, y_0, z_0) = (3, 2, 1)$ we get the symmetric form equations for the line $\dfrac{x - 3}{1} = \dfrac{y - 2}{3} = \dfrac{z - 1}{5}$.

2.3 Problems

1. Find a point-parallel equation for the line l that passes through the point P and is parallel to the vector $\mathbf{v}$. Find also a set of parametric equations for the line.
    a. P = (1, 2, 3) $\mathbf{v}$ = (2, 1, 4)   b. P = (3, 1, 3) $\mathbf{v}$ = (4, 2, 5)
    c. P = (4, 1, 5) $\mathbf{v}$ = (1, 0, 3)   d. P = (1, –2, 2) $\mathbf{v}$ = (2, 3, –1)
    e. P = (2, 0, –1) $\mathbf{v}$ = (1, –2, 3)   f. P = (–3, 2, 0) $\mathbf{v}$ = (0, –1, 3)

2. Find a set of parametric equations for the line l that passes through points P and Q.
    a. P = (1, 3, 4)   Q = (2, 5, 1)   b. P = (4, 3, –1)   Q = (2, 2, 3)
    c. P = (2, –1, –3)   Q = (1, 0, –2)   d. P = (5, 1, 3)   Q = (1, –1, 2)
    e. P = (4, 0, –5)   Q = (1, 3, –4)   e. P = (2, 2, –2)   Q = (3, –3, 0)

3. Find a set of symmetric equations for each of the lines of question 2 above.

2.4   ANGLES BETWEEN OBJECTS IN R³

In this section we will look at what is meant by the angle between two planes in $\mathbf{R}^3$ and by the angle between two lines in $\mathbf{R}^3$. We will use vector methods to find these angles.

The angle between two planes
Two planes that are not parallel to each other will intersect each other. The angle between the two planes is equal to the angle between their normals. Unless the intersection is a right angle, two distinct angles between the planes are created, an acute angle θ and the supplementary angle $180° - \theta$. In this case we will use the smaller acute angle as the angle between the two planes and refer to it as the **dihedral angle** between the planes. [Dihedral comes from the Greek words *dis* meaning two and *hedra* meaning face.]
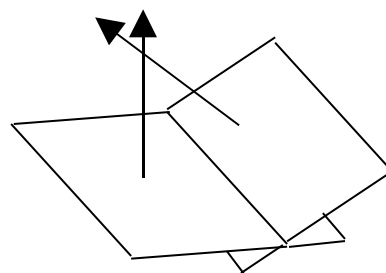
Example
Find the dihedral angle between the planes having equations $x + 2y + 3z = 4$ and $3x - 4y + 2z = 7$.

Solution

A pair of normals to the given planes are $\mathbf{n_1} = (1, 2, 3)$ and $\mathbf{n_2} = (3, -4, 2)$. The cosine of the angle between $\mathbf{n_1}$ and $\mathbf{n_2}$ is

$$\cos\theta = \frac{\mathbf{n_1} \bullet \mathbf{n_2}}{\|\mathbf{n_1}\| \|\mathbf{n_2}\|} = \frac{(1, 2, 3) \bullet (3, -4, 2)}{\|(1, 2, 3)\| \|(3, -4, 2)\|} = \frac{3-8+6}{\sqrt{1+4+9}\sqrt{9+16+4}} = \frac{1}{\sqrt{14}\sqrt{29}}$$

Using a calculator we find the dihedral angle is $\theta = \cos^{-1}\left(\frac{1}{\sqrt{14}\sqrt{29}}\right) = 87°$. Since this angle is less than $90°$ it is the required dihedral angle.

Example

Find the dihedral angle between the planes having equations $2x + 3y - 4z = 5$ and $4x - 2y + 3z = 6$.

Solution

A pair of normals to the given planes are $\mathbf{n_1} = (2, 3, -4)$ and $\mathbf{n_2} = (4, -2, 3)$. The cosine of the angle between $\mathbf{n_1}$ and $\mathbf{n_2}$ is

$$\cos\theta = \frac{\mathbf{n_1} \bullet \mathbf{n_2}}{\|\mathbf{n_1}\| \|\mathbf{n_2}\|} = \frac{(2, 3, -4) \bullet (4, -2, 3)}{\|(2, 3, -4)\| \|(4, -2, 3)\|} = \frac{8-6-12}{\sqrt{4+9+16}\sqrt{16+4+9}} = \frac{-10}{\sqrt{29}\sqrt{29}} = -\frac{10}{29}$$

Using a calculator we find the angle $\theta = \cos^{-1}\left(-\frac{10}{29}\right) = 110°$. Since this angle is greater than $90°$, the desired dihedral angle is $180° - 110° = 70°$.

Skew lines

Two noncoincident lines in $\mathbf{R}^2$ are either parallel and have no points in common or they intersect in exactly one point. In $\mathbf{R}^3$ there is another possibility. Two noncoincident lines may be nonparallel and nonintersecting. Such a pair of lines are said to be **skew lines**. The adjoining diagram illustrates a pair of skew lines that do not intersect since they lie in parallel planes, but the lines themselves are not parallel. We define the angle between skew lines to be the angle between a pair of intersecting lines that are parallel to the skew lines.
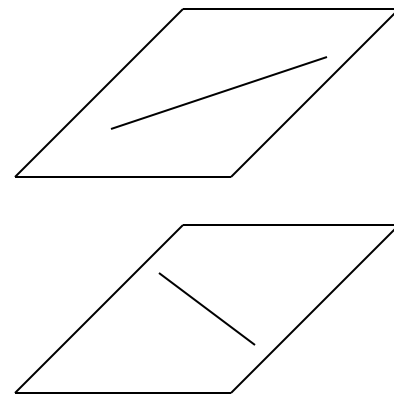
When two lines intersect, if the intersection is not at right angles two distinct angles are formed, an acute angle θ and its supplement $180° - \theta$. We will use the acute angle to describe the angle between the two lines.

Example

Find the angle between the skew lines l_1: $x = 2 + 3t$, $y = 4 - t$, $z = 3 + 2t$ and
l_2: $x = 1 - t$, $y = 5 + 2t$, $z = 6 + 3t$.

Solution

The vector $\mathbf{v}_1 = (3, -1, 2)$ is parallel to line l_1 and $\mathbf{v}_2 = (-1, 2, 3)$ is parallel to line l_2. The angle between the lines is equal to the angle between the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Let θ be this angle. Then $\cos\theta = \dfrac{\mathbf{v}_1 \bullet \mathbf{v}_2}{\|\mathbf{v}_1\|\,\|\mathbf{v}_2\|} = \dfrac{(3,-1,2)\bullet(-1,2,3)}{\|(3,-1,2)\|\,\|(-1,2,3)\|} = \dfrac{-3-2+6}{\sqrt{9+1+4}\sqrt{1+4+9}} = \dfrac{1}{14}$ and so

the required angle is $\theta = \cos^{-1}\left(\dfrac{1}{14}\right) = 86°$.

2.4   Problems

1. Find the dihedral angle between the following pairs of planes.
    a. $2x + 3y - z = 5$ $x + y + 5z = 2$ b. $x + 2y + 3z = 4$ $-2x + 3y + z = 1$
    c. $x + y = 1$ $7x + 7z = 2$ d. $2x + y + z = 3$ $2x + 2y = 5$
    e. $2x + 2y = 3$ $2x + y + 2z = 5$ f. $x + z = 5$ $x + 4y + 8z = 12$

2. Find the cosine of the angle between the pairs of skew lines.
    a. $x = 1 + t$, $y = 2 - 3t$, $z = 5 + t$ and $x = 3 - 2t$, $y = 5 + 4t$, $z = 1 - t$.
    b. $x = 7 + 3t$, $y = 1 - t$, $z = 2 + 2t$ and $x = 2 + t$, $y = 3 - 2t$, $z = 1 + 5t$
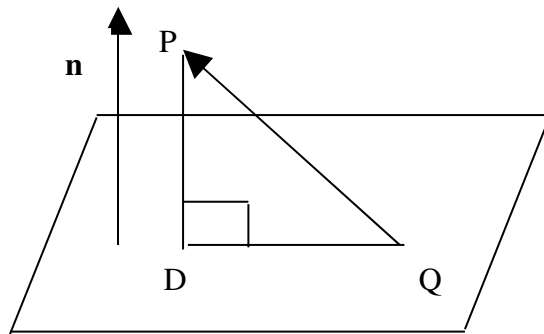   c. $\dfrac{x-3}{2} = \dfrac{y+1}{2} = \dfrac{z+2}{1}$ and $\dfrac{x-1}{3} = \dfrac{y+3}{3} = \dfrac{z+5}{2}$
    d. $\dfrac{x-5}{4} = \dfrac{y+3}{-1} = \dfrac{z+2}{-2}$ and $\dfrac{x-2}{5} = \dfrac{y+2}{-1} = \dfrac{z+3}{3}$

2.5   DISTANCES IN $\mathbf{R}^3$

Vector methods developed in chapter one will be used in this section to find a variety of distances in $\mathbf{R}^3$. We will find the distance between a point and a plane; between a point and a line; and between two lines in $\mathbf{R}^3$.

Distance between a point and a plane

Let P be the given point and let the plane have the equation $ax + by + cz = d$. A normal vector to the plane is $\mathbf{n} = (a, b, c)$. Let Q be any point on the plane. The distance between the point P and the plane is the magnitude of the projection of $\overrightarrow{QP}$ on $\mathbf{n}$.

Now $\text{proj}_\mathbf{n} \overrightarrow{QP} = \dfrac{\overrightarrow{QP} \bullet \mathbf{n}}{\mathbf{n} \bullet \mathbf{n}} \mathbf{n}$ and so the distance $= \left| \text{proj}_\mathbf{n} \overrightarrow{QP} \right| =$

$$\left| \frac{\overrightarrow{QP} \bullet \mathbf{n}}{\mathbf{n} \bullet \mathbf{n}} \mathbf{n} \right| = \frac{\left| \overrightarrow{QP} \bullet \mathbf{n} \right|}{\|\mathbf{n}\| \|\mathbf{n}\|} \|\mathbf{n}\| = \frac{\left| \overrightarrow{QP} \bullet \mathbf{n} \right|}{\|\mathbf{n}\|} = \frac{\left| \overrightarrow{PQ} \bullet \mathbf{n} \right|}{\|\mathbf{n}\|}. \qquad \left[\text{Since } \left| \overrightarrow{QP} \bullet \mathbf{n} \right| = \left| \overrightarrow{PQ} \bullet \mathbf{n} \right|. \right]$$

Example

Find the distance between the point $P = (2, 1, 3)$ and the plane $2x + y + 2z = 3$.

Solution

An arbitrary point Q on the plane is $Q = (1, 1, 0)$. Then $\overrightarrow{QP} = (2, 1, 3) - (1, 1, 0) = (1, 0, 3)$ and a normal vector to the plane is $\mathbf{n} = (2, 1, 2)$.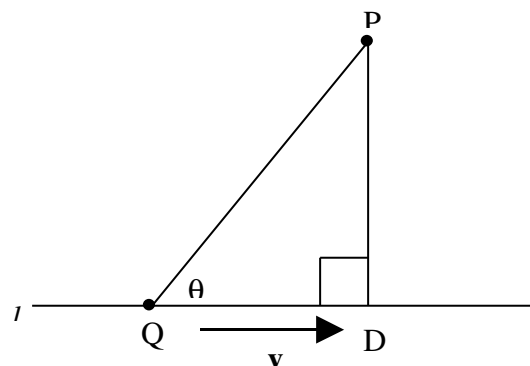 The distance between the point P and the plane is therefore equal to $\dfrac{\left| \overrightarrow{QP} \bullet \mathbf{n} \right|}{\|\mathbf{n}\|} = \dfrac{\left| (1, 0, 3) \bullet (2, 1, 2) \right|}{\|(2, 1, 2)\|} = \dfrac{|2 + 0 + 6|}{\sqrt{4 + 1 + 4}} = \dfrac{8}{3}$.

Distance between a point and a line in $\mathbf{R}^3$

Let l be a line passing through a point Q and parallel to the vector $\mathbf{v}$. Let P be a point not on the line l. The distance between P and l is the perpendicular distance of the point P from l. In the adjoining diagram the distance between P and l is the length of the line segment PD. Let θ be the angle between $\mathbf{v}$ and $\overrightarrow{QP}$. Then the required distance is

$$\left\| \overrightarrow{PD} \right\| = \left\| \overrightarrow{QP} \right\| \cdot \sin\theta = \frac{\left\| \overrightarrow{QP} \right\| \cdot \|\mathbf{v}\| \cdot \sin\theta}{\|\mathbf{v}\|} = \frac{\left\| \overrightarrow{QP} \times \mathbf{v} \right\|}{\|\mathbf{v}\|} = \frac{\left\| \overrightarrow{PQ} \times \mathbf{v} \right\|}{\|\mathbf{v}\|}$$

Example

Find the distance between the point $P = (5, 7, 10)$ and the line having parametric equations $x = 2 + 3t, \; y = 3 - 3t, \; z = 4 + t$.
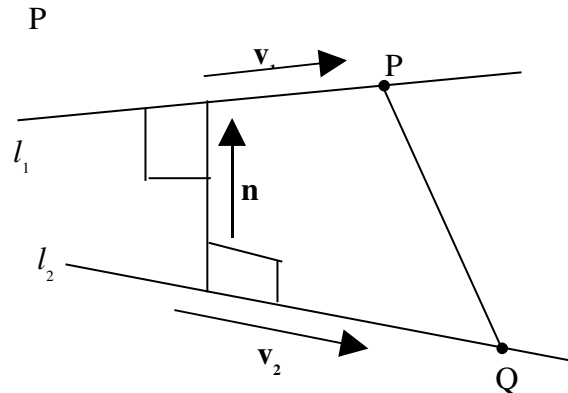
Solution

The vector $\mathbf{v} = (3, -3, 1)$ is a vector parallel to the given line. The point $Q = (2, 3, 4)$ is a point on the line so $\overrightarrow{QP} = (5, 7, 10) - (2, 3, 4) = (3, 4, 6)$.

Also $\overrightarrow{QP} \times \mathbf{v} = \left(\begin{vmatrix} 4 & 6 \\ -3 & 1 \end{vmatrix}, -\begin{vmatrix} 3 & 6 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 3 & 4 \\ 3 & -3 \end{vmatrix} \right) = (22, 15, -21)$. The required distance is equal

to $\dfrac{\left\| \overrightarrow{QP} \times \mathbf{v} \right\|}{\left\| \mathbf{v} \right\|} = \dfrac{\left\| (22, 15, -21) \right\|}{\left\| (3, -3, 1) \right\|} = \dfrac{\sqrt{484 + 225 + 441}}{\sqrt{9 + 9 + 1}} = \dfrac{\sqrt{1150}}{\sqrt{19}} = \dfrac{25\sqrt{46}}{\sqrt{19}}$.

The distance between two lines in $\mathbf{R}^3$

Let l_1 be a line through P that is parallel to the vector $\mathbf{v_1}$. Let l_2 be a line through Q that is parallel to the vector $\mathbf{v_2}$. The vector $\mathbf{n} = \mathbf{v_1} \times \mathbf{v_2}$ is a normal vector to both lines l_1 and l_2. The distance between the lines is the magnitude of the projection of $\overrightarrow{QP}$ on $\mathbf{n}$. If θ is the angle between $\overrightarrow{QP}$ and $\mathbf{n}$, then the distance between the lines is equal to

$$\left\| \overrightarrow{QP} \right\| \cdot |\cos\theta| = \dfrac{\left\| \overrightarrow{QP} \right\| \cdot \left\| \mathbf{n} \right\| \cdot |\cos\theta|}{\left\| \mathbf{n} \right\|} = \dfrac{\left| \overrightarrow{QP} \bullet \mathbf{n} \right|}{\left\| \mathbf{n} \right\|} = \dfrac{\left| \overrightarrow{PQ} \bullet \mathbf{n} \right|}{\left\| \mathbf{n} \right\|}.$$

Example

Find the distance between the lines l_1: $x = 3 + 2t$, $y = 5 - 3t$, $z = 4 + 4t$ and
l_2: $x = 1 + t$, $y = 2 + 2t$, $z = 3 - 3t$.

Solution

$P = (3, 5, 4)$ is a point on line l_1 and $Q = (1, 2, 3)$ is a point on line l_2. The vector from Q to P is $\overrightarrow{QP} = (3, 5, 4) - (1, 2, 3) = (2, 3, 1)$. The vector $\mathbf{n_1} = (2, -3, 4)$ is parallel to line l_1 and the vector $\mathbf{n_2} = (1, 2, -3)$ is parallel to line l_2. The normal vector to lines l_1 and l_2 is

the vector $\mathbf{n} = \mathbf{n_1} \times \mathbf{n_2} = \left(\begin{vmatrix} -3 & 4 \\ 2 & -3 \end{vmatrix}, -\begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix}, \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} \right) = (1, 10, 7)$. The distance between

the lines is $\dfrac{\left| \overrightarrow{QP} \bullet \mathbf{n} \right|}{\left\| \mathbf{n} \right\|} = \dfrac{\left| (2, 3, 1) \bullet (1, 10, 7) \right|}{\left\| (1,10,7) \right\|} = \dfrac{2 + 30 + 7}{\sqrt{1 + 100 + 49}} = \dfrac{39}{\sqrt{150}} = \dfrac{39}{5\sqrt{6}}$.

2.5   Problems

1. Find the distance between the given point P and the given plane.
   a. $P = (1, 3, 2)$ $2x + 2y + z = 1$ b. $P = (3, 5, 4)$ $3x + 4y + 5z = 12$
   c. $P = (1, 0, 2)$ $2x + y + 2z = 1$ d. $P = (2, 2, 1)$ $3x + y + z = 9$

2. Find the distance between the following pairs of parallel planes by selecting a point on one plane and then finding the distance between that point and the other plane.
   a. $2x + 2y + z = 1$, $2x + 2y + z = 6$ b. $3x + y + z = 9$, $3x + y + z = 25$

3. Find the distance between the point P and the line l.
   a. $P = (1, 3, 4)$ l: $x = 4 + 3t$, $y = 5 + 2t$, $z = 6 + 3t$.
   b. $P = (5, 6, 4)$ l: $x = 1 + t$, $y = 2 - t$ $z = 2 + 2t$
   c. $P = (3, 2, 5)l$: $x = 1 - t$, $y = t$ $z = 2 + t$
   d. $P = (2, 3, 1)$ l: $x = 5 + 2t$ $y = 6 + t$ $z = 4 - t$

4. Find the distance between the two given lines.
   a. $\mathbf{x} = (1, 3, 2) + t(2, 5, 3)$ and $\mathbf{x} = (3, 5, 4) + t(1, 0, 1)$
   b. $x = 4 + 3t$, $y = 5 + 2t$, $z = 6 + 3t$ and $x = 1 + t$, $y = 2 - t$ $z = 2 + 2t$
   c. $\dfrac{x-3}{2} = \dfrac{y+1}{2} = \dfrac{z+2}{1}$ and $\dfrac{x-1}{3} = \dfrac{y+3}{3} = \dfrac{z+5}{2}$
   d. $\dfrac{x-5}{4} = \dfrac{y+3}{-1} = \dfrac{z+2}{-2}$ and $\dfrac{x-2}{5} = \dfrac{y+2}{-1} = \dfrac{z+3}{3}$

5. The line l is parallel to the given plane. Find the distance between the line and the plane.
   a. l: $\mathbf{x} = (1, 3, 2) + t(2, 5, 3)$ $x - y + z = 1$
   b. l: $\mathbf{x} = (3, 5, 4) + t(1, 0, 1)$ $2x + 3y - 2z = 3$
   c. l: $x = 1 + t$, $y = 2 - t$, $z = 2 + 2t$ $3x + 5y + z = 2$
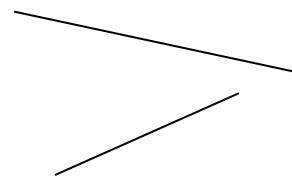
2.6   INTERSECTIONS

Two distinct planes that are not parallel intersect in a line (diagram A).   A line and a plane that are not parallel will intersect in a point (diagram B). Two nonparallel lines in $\mathbf{R}^3$ may or may not intersect. If they intersect, the intersection is a point (diagram C).
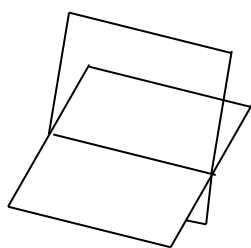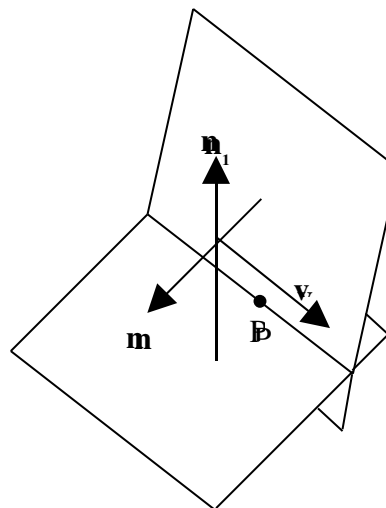
A
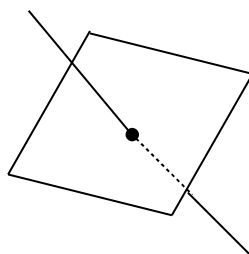
B

C

The line of intersection of two planes

Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be normals to a pair of intersecting planes. Then the vector $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$ is parallel to the line of intersection of the two planes. If P is any point on the line of intersection, then we can use the point-parallel form equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ to get an equation of the line of intersection.

$\mathbf{n}_1$

$\mathbf{v}$

$\mathbf{m}$

P

Example

Find a point-parallel form equation for the line of intersection of the two planes $2x + 3y + 4z = 1$ and $x - 5y + 6z = 3$. Also, find a set of parametric equations for the line of intersection of the two planes.

Solution

The planes have normals $\mathbf{n_1} = (2, 3, 4)$ and $\mathbf{n_2} = (1, -5, 6)$. The vector $\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2}$ is parallel to the required line. $\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2} = \left(\begin{vmatrix} 3 & 4 \\ -5 & 6 \end{vmatrix}, -\begin{vmatrix} 2 & 4 \\ 1 & 6 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 1 & -5 \end{vmatrix} \right) = (38, -8, -13)$. To find a point P on the line we must find a point that satisfies the two equations $2x + 3y + 4z = 1$ and $x - 5y + 6z = 3$. Since there are two equations with three variables we set $z = 0$ in the equations to get $2x + 3y = 1$
$$\text{and} \quad x - 5y = 3.$$
Solving for x and y we get $x = \dfrac{14}{13}$ and $y = -\dfrac{5}{13}$. Combining these values with $z = 0$ we get the point $P = \left(\dfrac{14}{13}, -\dfrac{5}{13}, 0 \right)$ so the point-normal form equation is

$$\mathbf{x} = \left(\frac{14}{13}, -\frac{5}{13}, 0 \right) + t(38, -8, -13).$$

The parametric equations are $x = \dfrac{14}{13} + 38t, \quad y = -\dfrac{5}{13} - 8t, \quad z = 0 - 13t$.

The intersection of a plane and a line

To find the point of intersection of a line and a plane we use the parametric form of the equation of the line and the standard form equation for the plane.   Substitute the values for x, y and z from the line equations into the equation of the plane and solve for the parameter. Now substitute the value of the parameter in the line equations to find the point of intersection. The next example illustrates the procedure.
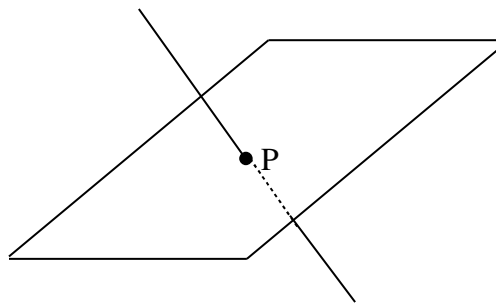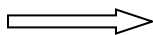

Example

Find the point of intersection of the line $x = 1 + t, \ y = 2 - t, \ z = 3 + 2t$ and the plane $4x + y - z = 6$.

Solution

$4x + y - z = 6$ $\Longrightarrow$ $4(1 + t) + (2 - t) - (3 + 2t) = 6$

$\Longrightarrow$ $4 + 4t + 2 - t - 3 - 2t = 6$ $\Longrightarrow$ $t = 3.$

Substituting $t = 3$ into the line equations gives $x = 1 + 3 = 4$, $y = 2 - 3 = -1$, $z = 3 + 6 = 9$. The point of intersection is therefore, $P = (4, -1, 9)$.

The intersection of two lines

To determine whether two lines in $\mathbf{R}^3$ have a point in common we equate the corresponding parts of their parametric equations and solve for the parameters. There will be three equations with two variables (parameters). If this system does not have a solution, the lines do not intersect. If the system has a solution, substituting the values of the parameters into the parametric equations gives the point of intersection.

Example

Find the point of intersection of the lines l_1: $x = 1 + 2t$, $y = 3 + t$, $z = 2 + 3t$ and l_2: $x = 1 + s$, $y = 2 + s$, $z = 1 + 2s$.

Solution

$$1 + 3t = 1 + s$$
$$3 + t = 2 + s$$
$$2 + 3t = 1 + 2s$$

$\Longrightarrow$

$$2t - s = 0 \quad (1)$$
$$t - s = -1 \quad (2)$$
$$3t - 2s = -1 \quad (3)$$

Solving equations (1) and (2) for t and s gives $t = 1$ and $s = 2$.

These values also satisfy equation (3), so the system has a solution and the point of intersection is found by substituting $t = 1$ (or $s = 2$) into the line equations. Thus for example if we substitute the value $t = 1$ into the equations for the first line we get $x = 1 + 2(1) = 3$, $y = 3 + 1 = 4$, $z = 2 + 3(1) = 5$. The point of intersection is (3, 4, 5). [Note that if instead we substitute $s = 2$ into the equations for the second line we will get $x = 1 + 3 = 3$, $y = 2 + 2 = 4$, $z = 1 + 2(2) = 5$ and so find (3, 4, 5) is the point of intersection as before.]

Example

Show that the lines $x = 1 + 2t, \; y = 3 + t, \; z = 2 + 3t$

and $x = 1 + s, \; y = 2 + s, \; z = 1 + 3s$ do not intersect.

Solution

$1 + 2t = 1 + s$ $\qquad\qquad\qquad\Longrightarrow\qquad\qquad$ $2t - s = 0$ $\qquad$ (1)

$3 + t = 2 + s$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $t - s = -1$ $\qquad$ (2)

$2 + 3t = 1 + 3s$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $3t - 3s = -1$ $\qquad$ (3)

Solving equations (1) and (2) gives $t = 1$ and $s = 2$. However, substituting $t = 1$ and $s = 2$ into the left hand side of equation (3) gives $3(1) - 3(2) = -3 \neq -1$, so equation (3) is not satisfied and hence there is no point of intersection.

2.6 Problems

1. Find a set of parametric equations for the line of intersection of the following pairs of planes.

   a. $x + 2y - 3z = 4$ and $2x - y + z = 1$ $\qquad\qquad$ b. $3x + 2y - z = 1$ and $4x + 2y + z = 3$

   c. $3x + y - z = 5$ and $x + y + 2z = 3$ $\qquad\qquad$ d. $x - y + 2z = 4$ and $2x - 3y + 2z = 1$

2. Find the coordinates of the point of intersection of the given line and plane.

   a. $x = 2 + t, \; y = 1 + t, \; z = 2 - t$ $\qquad$ $x + y + z = 6$

   b. $x = 4 - t, \; y = -2 + 2t, \; z = 3 - t$ $\qquad\qquad$ $2x + y + z = 7$

   c. $x = -2 + t, \; y = 3 - t, \; z = -1 + 4t$ $\;\; 2x + 3y + z = 1$

   d. $\mathbf{x} = (1, 4, 1) + t(2, 1, 2)$ $\qquad\qquad$ $(1, 1, -1) \bullet \mathbf{x} = 4$

3. Determine whether or not the following pairs of lines intersect. If they do intersect, find the coordinates of the point of intersection.

   a. $x = 2 + t, \; y = 1 + t, \; z = 2 - t$ $\qquad$ and $\qquad$ $x = 6 - s, \; y = 5 - s, \; z = 7 - 2s$

   b. $x = -2 + t, \; y = 3 - t, \; z = -1 + t$ $\qquad$ and $\qquad$ $x = 5 - s, \; y = 6 - s, \; z = 11 - 2s$

   c. $x = 4 - t, \; y = -2 + 2t, \; z = 3 - t$ $\qquad$ and $x = 1 + s, \; y = 2 - s, \; z = 4 - s$

d. $x = 2 + t, \ y = 1 + t, \ z = 2 - t$ and $x = 5 + s, \ y = 4 - s, \ z = 1 + 2s$

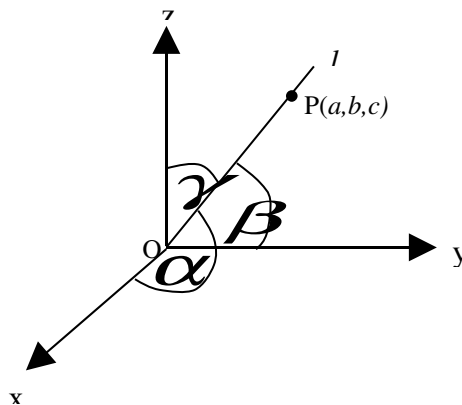## 2.7   DIRECTION NUMBERS AND DIRECTION COSINES

The orientation of a line l in $\mathbf{R}^3$ has been specified by giving a vector $\mathbf{v} = (v_1, v_2, v_3)$ that is parallel to the line l.   Since $c\mathbf{v} = (cv_1, cv_2, cv_3)$ is also parallel to l we see that any ordered set of three numbers (a, b, c) that are proportional to (v_1, v_2, v_3) will suffice to give the orientation of the line l.   We call any ordered set of three numbers (a, b, c) that are proportional to (v_1, v_2, v_3) a set of **direction numbers** for the line l. If we are given a set of direction numbers for a line we know its orientation in space.

Since the orientation of a line l in $\mathbf{R}^3$ is not changed if we move it to a new location but keep it parallel to the direction of the line l in its original position, we can restrict any further discussion to lines passing through the origin.
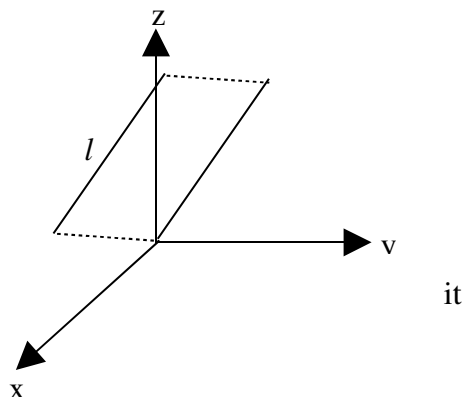
Direction angles

Let α, β and γ be the angles that the line l passing through the origin makes with the positive x-axis, positive y-axis and positive z-axis respectively. The angles α, β and γ are called a set of **direction angles** for the line l.   The orientation of a line in space can be specified by giving the direction angles instead of a set of direction numbers.

Direction cosines

If the point P $= (a, b, c)$ is on the line l then the vector $\overrightarrow{\text{OP}}$ is equal to

$$\overrightarrow{\text{OP}} = (a, b, c) - (0, 0, 0) = (a, b, c) .$$

Since $\overrightarrow{OP}$ lies on l, the set of numbers (a, b, c) are a set of direction numbers for line l. If we now let $d = \left\| \overrightarrow{OP} \right\|$, then the cosines of the three direction angles α, β, γ are

$$\cos\alpha = \frac{a}{d}, \quad \cos\beta = \frac{b}{d} \text{ and } \cos\gamma = \frac{c}{d}.$$ These cosines are proportional to a, b, c and form a special set of direction numbers for the line l called the **direction cosines** of l.

Theorem If $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of a line l, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Proof Let P $= (a, b, c)$ and let $d = \left\| \overrightarrow{OP} \right\|$.

Then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = (a/d)^2 + (b/d)^2 + (c/d)^2 = \dfrac{a^2 + b^2 + c^2}{d^2} = \dfrac{d^2}{d^2} = 1$

Example

The line l passes through the points P $= (1, 3, 4)$ and Q $= (5, 4, 7)$. Find a set of direction numbers and a set of direction cosines for the line l.

Solution

$\overrightarrow{PQ} = (5, 4, 7) - (1, 3, 4) = (4, 1, 3)$ so a set of direction numbers for l is $(4, 1, 3)$.

Since $d = \left| \overrightarrow{PQ} \right| = \sqrt{4^2 + 1^2 + 3^2} = \sqrt{26}$, a set of direction cosines for the line l is

$$\cos\alpha = \frac{4}{\sqrt{26}}, \quad \cos\beta = \frac{1}{\sqrt{26}}, \quad \cos\gamma = \frac{3}{\sqrt{26}}.$$

2.7 Problems

1. Find a set of direction numbers and a set of direction cosines for the line through points P and Q.

a. $P = (1, 2, 3)$ $Q = (3, 5, 4)$ b. $P = (3, 5, 4)$ $Q = (7, 6, 5)$
c. $P = (2, 2, 1)$ $Q = (3, 4, 4)$ d. $P = (5, 6, 7)$ $Q = (4, 3, 2)$

2. Find a set of direction numbers and direction cosines for the following lines.
   a. $x = 1 + t,\ y = 2 - 3t,\ z = 3 + 4t$ b. $x = 2 - t,\ y = 3 + 2t,\ z = 4$
   c. $\mathbf{x} = (2,3,1) + t(4,2,3)$ d. $\mathbf{x} = (5,1,3) + t(3,0-4)$

3. Find a set of parametric equations for the line passing through the given point P and having the given set of direction numbers.
   a. $P = (2, 5, 1)$ $(a, b, c) = (3, 2, 4)$ b. $P = (3, -2, 1)$ $(a,\ b,\ c) = (-1,\ 0\ 2)$
   c. $P = (4, 1, 3)(a, b, c) = (3, 0, 2)$ d. $P = (5, 0, -2)$ $(a, b, c) = (2, 1, -2)$

4. Find a standard form equation for the plane passing through the point P and having a normal with the given set of direction numbers.
   a. $P = (2, 5, 1)$ $(a, b, c) = (3, 2, 4)$ b. $P = (3, -2, 1)$ $(a,\ b,\ c) = (-1,\ 0\ 2)$
   c. $P = (4, 1, 3)(a, b, c) = (3, 0, 2)$ d. $P = (5, 0, -2)$ $(a, b, c) = (2, 1, -2)$