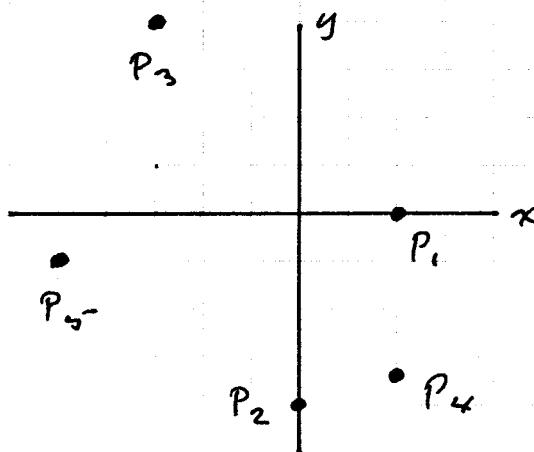
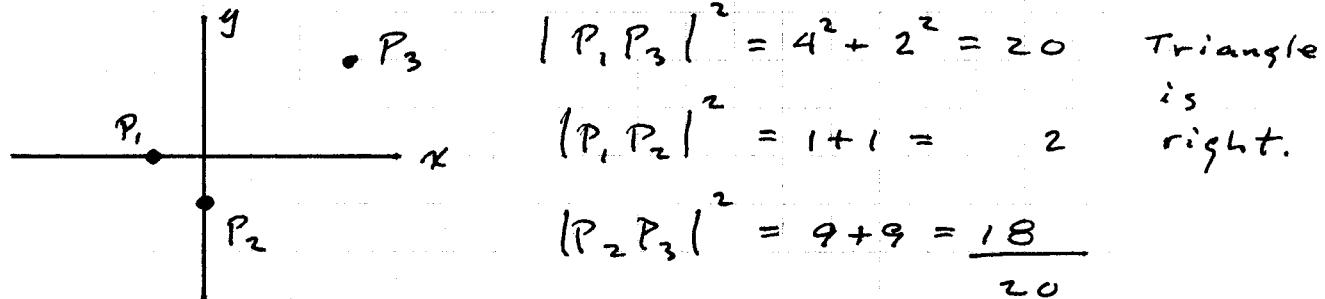


# Exercises for MATH1210 Supplementary Notes

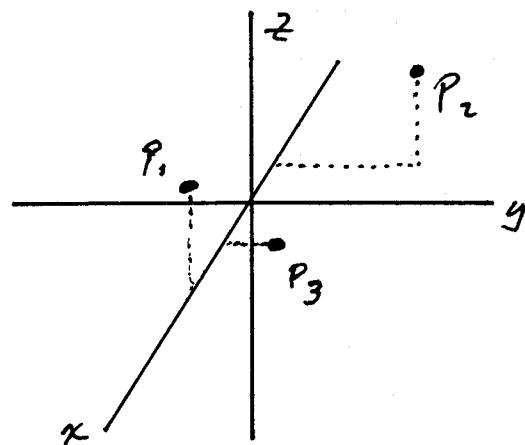
1.1.



2.



3.



$$|P_1 P_2|^2 = 9 + 9 = 18$$

$$|P_1 P_2| = 3\sqrt{2}$$

$$|P_3 P_1|^2 = 1 + 1 + 4 = 6$$

$$|P_3 P_1| = \sqrt{6}$$

$$|P_3 P_2|^2 = 4 + 4 + 4 = 12$$

$$|P_3 P_2| = 3\sqrt{2}$$

$$|P_1 P_3|^2 + |P_2 P_3|^2 = |P_1 P_2|^2; \text{ so the triangle is right.}$$

$$2.1(a) \overrightarrow{P_1 P_2} + \overrightarrow{P_1 P_4} = [1, -1, 3] + [-2, 2, 2]$$

$$= [-1, 1, 5]$$

$$\overrightarrow{P_1 P_3} = [-1, 1, 5] = \overrightarrow{P_1 P_2} + \overrightarrow{P_1 P_4}$$

$P_1 P_2 P_3 P_4$  is a parallelogram.

$$(b) \|\overrightarrow{P_1 P_2}\| = \sqrt{1+1+9} = \sqrt{11} = \|\overrightarrow{P_3 P_4}\|$$

$$\|\overrightarrow{P_1 P_4}\| = \sqrt{4+4+4} = \sqrt{12} = \|\overrightarrow{P_3 P_2}\|$$

Let the angle between  $\overrightarrow{P_1 P_2}$  and  $\overrightarrow{P_1 P_4}$  be  $\theta$  and between  $\overrightarrow{P_3 P_2}$  and  $\overrightarrow{P_3 P_4}$  be  $\phi$ .

$$\cos \theta = \frac{\overrightarrow{P_1 P_2} \cdot \overrightarrow{P_1 P_4}}{\|\overrightarrow{P_1 P_2}\| \|\overrightarrow{P_1 P_4}\|} = \frac{-2-2+6}{\sqrt{11} \sqrt{12}} = \frac{2}{\sqrt{132}}$$

$$\overrightarrow{P_3 P_4} = [-1, 1, -3]. \quad \overrightarrow{P_3 P_2} = [2, -2, -2]$$

$$\cos \phi = \frac{\overrightarrow{P_3 P_4} \cdot \overrightarrow{P_3 P_2}}{\|\overrightarrow{P_3 P_4}\| \|\overrightarrow{P_3 P_2}\|} = \frac{-2-2+6}{\sqrt{11} \sqrt{12}} = \frac{2}{\sqrt{132}}.$$

Since the cosines of  $\theta$  and  $\phi$  are equal,  $\theta$  and  $\phi$  are equal.

(c) Let the angle between  $\overrightarrow{P_1 P_2}$  and  $\overrightarrow{P_3 P_4}$  be  $\psi$ . Since  $\overrightarrow{P_1 P_2} = -\overrightarrow{P_3 P_4}$ , the angle is  $\pi$ .

$$2 \vec{u} + \vec{v} = \frac{1}{2\sqrt{2}} [1, 2, 1] + \frac{1}{2\sqrt{2}} [1, 0, -1]$$

$$= \frac{1}{2\sqrt{2}} [2, 2, 0] = \frac{1}{\sqrt{2}} [1, 1, 0]$$

$$\vec{u} - \vec{v} = \frac{1}{2\sqrt{2}} [0, 2, 2] = \frac{1}{\sqrt{2}} [0, 1, 1]$$

Both vectors have length 1. The cosine of the angle between them is  $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = \frac{1}{2}$ . The angle is therefore  $\pi/3$ .

$$2.3(a) \|\vec{a}\| = \sqrt{1+1+1} = \sqrt{3} \quad \|\vec{b}\| = \sqrt{1+2+1} = \sqrt{6}$$

$$\vec{a} \cdot \vec{b} = -1 + 2 + 1 = 2 \quad \cos \alpha = \frac{2}{\sqrt{3}\sqrt{6}} = \frac{\sqrt{2}}{3}.$$

$$(b) \vec{a} = [1, 1, 1] \quad \|\vec{a}\| = \sqrt{3} \quad \|\vec{b}\| = \sqrt{6}$$

$$\vec{b} = [-1, 2, 1] \quad \vec{a} \times \vec{b} = [1-2, -1-1, 2+1] \\ = [-1, -2, 3]$$

$$\|\vec{a} \times \vec{b}\| = \sqrt{1+4+9} = \sqrt{14}.$$

$$\sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \frac{4}{9}} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}.$$

$$\|\vec{a}\| \|\vec{b}\| \sin \alpha = \sqrt{3} \sqrt{6} \frac{\sqrt{5}}{3} = \sqrt{14} = \|\vec{a} \times \vec{b}\|.$$

$$4. \hat{i} \times \hat{j} = \hat{k}, \hat{k} \times \hat{j} = -\hat{i}, -\hat{i} \times \hat{j} = -\hat{k} = [0, 0, 1].$$

$$\{0, 0, 1\} \cdot \{1, 2, 7\} = -7.$$

$$5(a) \|\vec{r}\| = \sqrt{4+9+3} = \sqrt{16} = 4.$$

$$\hat{r} = \left[ \frac{1}{2}, \frac{3}{4}, \frac{\sqrt{3}}{4} \right].$$

$$(b) 2\hat{r} = [1, \frac{3}{2}, \frac{\sqrt{3}}{2}]$$

(c) Length cannot be negative although a directed distance can.

$$6(a) \vec{p} - \vec{g} = \vec{op} - \vec{oq} \stackrel{(3.5)}{=} \vec{op} + (-1)\vec{qq} = \vec{op} + \vec{qo} \text{ (Remark 3.3)}$$

$$= \vec{qp} \text{ by Definition of Addition.}$$

$$(b) \vec{om} = \vec{op} + \frac{1}{2}\vec{pq} = \vec{p} + \frac{1}{2}(\vec{oq} - \vec{op}) = \vec{p} + \frac{1}{2}(\vec{q} - \vec{p}) = \frac{1}{2}(\vec{p} + \vec{q}).$$

$$(c) \vec{ot} = \vec{op} + \frac{1}{3}(\vec{pq}) = \vec{p} + \frac{1}{3}(\vec{oq} - \vec{op}) = \vec{p} + \frac{1}{3}(\vec{q} - \vec{p}) = \frac{2}{3}\vec{p} + \frac{1}{3}\vec{q}.$$

(d) Sphere with centre P and radius 2.

$$(e) \vec{op}_1 = \frac{1}{2}\vec{op}. \quad \vec{oq}_1 = \frac{1}{2}\vec{oq}.$$

$$\vec{q}_1 \vec{p}_1 = \frac{1}{2}\vec{op} - \frac{1}{2}\vec{oq} = \frac{1}{2}(\vec{op} - \vec{oq}) = \frac{1}{2}\vec{qp}.$$

7. Let the diagram on p. 6 of the Supplementary Notes be a cube with  $B = (1, 1, 1)$ . The angle to consider is between  $[1, 0, 0]$  and  $[1, 1, 1]$  or the unit vector  $\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$ .  $\cos \theta = [1, 0, 0] \cdot \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right] = \frac{1}{\sqrt{3}}$ .  $\theta$  is the angle with cosine  $\frac{1}{\sqrt{3}}$ , i.e.  $0.955$  approx.

2.8 Let the diagram on p. 6 represent the \$l\$-piped.  
 $B = (a, b, c)$  Unit vector in the direction of the diagonal is  $\frac{1}{\sqrt{a^2+b^2+c^2}} [a, b, c]$ . Cosines of the angle between the axes and the diagonal are

$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$ . They are called the direction cosines of the diagonal.

$$3.1(a) \vec{u} = [-2, 2, 3] \quad \vec{u} \times \vec{v} = [17, 11, 4]$$

$$\vec{v} = [1, -3, 4] \quad x = 1 + t(17),$$

Scalar parametric equations are  $y = 2 + t(11)$ ,  
 $z = -3 + t(4)$ .

(b) Let  $t = 2$ . Then  $x = 35$ ,  $y = 24$ ,  $z = 5$ .

$$(c) t = \frac{x-1}{17} = \frac{y-2}{11} = \frac{z+3}{4}$$

$$2(a) 17(x-1) + 11(y-2) + 4(z+3) = 0.$$

$$(b) \overrightarrow{OP_0} = [1, 2, -3]. -2\vec{u} = [4, -4, -6]$$

$$3\vec{v} = [3, -9, 12]. \overrightarrow{OP} = [8, -11, 3]$$

$$P = (8, -11, 3). (c) \text{ Check: } 17(7) + 11(-13) + 4(6) = 0.$$

3(a) The direction of the line of intersection of  $P_1$  and  $P_2$  must be  $\perp$  to their normal vectors  $[1, 1, -2]$  and  $[3, 4, 1]$ . Their cross product is  $[1, -7, 1]$ . A point on the line can be obtained by intersecting it with the plane  $z=0$ .

Substituting  $z=0$  into the original equations

gives 
$$\begin{cases} x+y=3 & \leftarrow x=7 \\ 3x+4y=5 \\ 3x+3y=9 \\ y=-4 \end{cases}$$
 The scalar equations of the line are

$$x = 7 + 9t$$

$$y = -4 - 7t$$

$$z = t.$$

An easier way to do this problem is coming.

3.3(b) Having their normal vectors parallel but not being multiples one of the other these equations represent distinct but parallel planes and so have no point of intersection.

- (c) The equations of the line can be substituted into the equation of the plane to determine the parameter  $t$  at the point of intersection.

$$\begin{aligned} 4 - 2t - 3 - t - 8 - 2t &= 3 \\ -5t &= 10 \\ t &= -2 \end{aligned}$$

The point is  $(8, -1, 2)$  from the equations for the line, and it satisfies the equation for the plane — an appropriate check.

- (d) The  $s$  and  $t$  expressions for the point of intersection (if any) can be solved for their values, which must give the same point in the equations for both lines

$$\begin{aligned} 4 - 2t &= x = 2 + s \\ -3 - t &= y = -4 + 2s \\ 4 + t &= z = 5 - s \end{aligned} \quad \left. \begin{array}{l} \text{enough} \\ \text{---} \\ \text{---} \end{array} \right\} \begin{aligned} -t - 2s &= -1 \\ t + s &= 1 \\ -s &= 0 \\ t &= 1 \end{aligned}$$

These values satisfy all three equations and give the point of intersection (twice) as  $(2, -4, 5)$ .

- (e) The same approach gives equations

$$\begin{aligned} 4 - 2t &= x = 6 - 2s \\ -3 - t &= y = -1 + s \\ 4 + t &= z = 5 - s \end{aligned} \quad \left. \begin{array}{l} \text{enough} \\ \text{---} \\ \text{---} \end{array} \right\} \begin{aligned} -2t + 2s &= 2 \\ -t - s &= 2 \\ -t + s &= 1 \\ -2t &= 3 \\ t &= -\frac{3}{2} \\ s &= -\frac{1}{2} \end{aligned}$$

Unlike the situation in part (d), these values do not satisfy the third equation.  $4+t=2\frac{1}{2}$ , but  $5-s=5\frac{1}{2}$ . These equations have no solution as one sees more easily if one tries the second and third. It is essential to check a supposed solution obtained from two equations in the third. There is no solution because the lines do not meet. Such lines are called skew.

(f) The same approach taken here

$$4-2t = x = 6-2s$$

$$-3-t = y = -1-s$$

$$4+t = z = 3+s$$

again leads either to a non-solution or to a contradiction  $-t+s=2$  depending on the equations you choose to use.

$$\begin{array}{r} -t+s=2 \\ t-s=-1 \\ \hline 0=1 \end{array}$$

Again the reason is that the lines do not intersect, but this time because they are parallel.

(Compare their direction vectors.)

(g) What about  $4-2t = x = 2-2s$

$$-3-t = y = -4-s$$

$$4+t = z = 5+s ?$$

Comparison of the direction vectors might lead one rashly to reject these as parallel lines, but solving shows that they are the same line, therefore having infinitely many common points. Solution is not superfluous.